A NOTE ON THE EFFECT OF WALL CONDUCTION ON HEAT TRANSFER IN COUETTE FLOW WITH AND WITHOUT INJECTION

V. K. TRIVEDI

Defence Science Laboratory, Delhi.

(Received 3 October 1969; revised 21 March 1970)

Problems of temperature distribution in transpiration cooled porous body and in hot gas stream have been solved in conjugation. The effect of wall conduction on heat transfer in Couette flow has been exhibited graphically and discussed.

The problem of heat transfer in a porous plate one side of which is in contact with a hot gas flow and through which a coolant is injected from the colder side has been studied by various authors. This problem has attracted much attention because of application of technique called transpiration cooling in aviation as well as chemical technology.

The study of temperature distribution in a transpiration cooled porous plate was first attempted by Grootenhuis & Moore[1]. Defining the porosity as the ratio of volume of voids to the total volume of the porous metal specimen and utilizing the idea of volume heat transfer coefficient they obtained a third degree differential equation. It was easily solved giving the temperature distribution in porous plate as well as in the coolant. The constants of solution were obtained by using known boundary conditions. Weinbaum & Wheeler[2] used a somewhat similar approach. Since the porous structure of the plate formed a very complicated three dimensional network, it was assumed to be equivalent to a simple network of identical cylindrical channels running parallel from one end of the specimen to the other. Energy balance in a small element gave rise to a third degree differential equation. The authors analysed the solution of this equation. One of their observations was that the temperature of the coolant and that of porous plate must be equal at every point except very very near the cold side. Many papers using this approximation have appeared since then, e. g. Green Leon & Calf Downey[3] and Mayer & Bartas[4]. This approximation reduces the differential equation to second degree only. But Grootenhuis[5] pointed out that an error of sign had crept in the analysis of Weinbaum & Wheeler[2] and hence this approximation was not well founded. A mathematically general solution of the three dimensional problem of gas flow in porous body and associated temperature distributions has been given by Bland[6]. Only in some very special cases his equations could be solved exactly. In all other cases a numerical solution was necessary.

The problem of temperature distribution in a hot gas flowing over an ordinary flat plate has been solved exactly by Schlichting[7] in some simple cases. He has also studied a porous flat plate with suction. Friedman[8] has studied the temperature distribution in the laminar zone of hot gas stream flowing past a porous plate with coolant injection and gave a relation between coolant flow and the temperature of hot side assuming a given heat transfer coefficient and Newton type heat transfer between the hot gas flow and the porous wall.

The problem of heat conduction in the hot gas stream and the porous wall have thus been treated as completely independent problems so far. It has been pointed out by Perelman[9] that in order to obtain exact estimates of the surface temperatures the problem of heat transfer in the gas should be solved in conjunction with the heat conduction problem in the solid body with conditions of continuity of flux and temperature at the interface. Such problems have been termed as conjugate problems. In this paper, therefore, we have discussed the problem of heat transfer in porous wall in conjunction with the heat flow in the hot gas side and have calculated the effect of heat conduction within the wall on the heat transfer in the gas flow. We have utilized the third degree equation of Grootenhuis for the temperature distribution in the porous plate. The hot gas flow has been assumed Couette flow, the velocity-profile being complicated due to the presence of injection. It has also been assumed that the characters of hot gas are not affected by coolant. The velocity profile thus obtained has been substituted in the energy equation of hot gas. The energy equation has now been solved in conjunction with the aforesaid third degree equation. To get the constants of the solution we have used the following boundary and interface conditions:

(a) known temperature of the free stream hot gas

(b) known temperature of the cold side of porous plate

(c) equality of temperatures of porous plate and hot gas stream at contact surface

(d) continuity of flux at the contact surface

(e) equality of temperatures of the porous plate and the coolant fluid at the contact surface i. e. $T=t$. (This condition was pointed out by Grootenhuis[5] and is based on the fact that within the porous wall, heat transfer is from the material to the coolant and therefore $t > T$ but outside this surface the coolant is hotter and is transferring heat to the surface and therefore $t < T$. Both these conditions can be satisfied only if, at the surface, the two temperatures are assumed to be equal).

NOMENCLATURE

t, θ and T — Temperature distributions in porous plate, hot gas stream and coolant fluid respectively

t_0 — Temperature of the injectant side of the porous plate

θ_∞ — Temperature of free stream of hot gas

U_∞ — Free stream velocity of hot gas

μ — Coefficient of viscosity of hot gas

η — Distance between the static and the moving plate i. e. thickness of the hot gas layer over porous plate

K_g — Conductivity of hot gas

K_s — Conductivity of porous material of the plate

C_p — Heat capacity of the hot gas

G — Mass of the coolant passing through unit area of porous plate per unit time

δ — Thickness of the porous plate

h' — Average volume heat transfer coefficient, i.e. amount of heat transferred from unit volume of porous body to the coolant per unit time when the difference of temperature between them is one degree centigrade

y — Distance co-ordinate perpendicular to hot gas flow

$$v_1 = \frac{t - t_0}{\theta_\infty - t_0}, \qquad v_2 = \frac{\theta - t_0}{\theta_\infty - t_0}, \qquad \xi_1 = \frac{y}{\delta},$$

$$\xi_2 = \frac{y}{\eta}, \qquad K = \frac{K_g}{K_s}, \qquad \lambda = \frac{\delta}{\eta}$$

$$St_c' = \frac{h'\delta}{GC_p}, \qquad Nu_c' = \frac{h'\delta^2}{K_g}, \qquad Re_g' = \frac{G\eta}{\mu}$$

$$E_g' = \frac{U^2_\infty}{(\theta_\infty - t_0)C_p}$$

$$Pr_g = \frac{C_p\mu}{K_g}, \qquad Pe_g' = Re_g' \times Pr_g$$

$$R = \sqrt{\left(\frac{St_c'}{2}\right)^2 + K\,Nu_c'} + \frac{St_c'}{2}$$

$$S = \sqrt{\left(\frac{St_c'}{2}\right)^2 + K\,Nu_c'} - \frac{St_c'}{2}$$

$$\Delta = \left\{ S^2 (1 - e^{R}) - R^2 (1 - e^{-S}) + \frac{RS\,(R + S)}{\lambda\,K\,Pe_g'} (1 - e^{Pe_g'}) \right\}$$

$$\pi = \left\{ 1 - \frac{Pr_g\,E_g}{2\,(Pr_g - 2)\,(1 - e^{Re_g'})^2} \left[(e^{2Re_g'} - 1) + \frac{2\,Re_g'\,(1 - e^{P\,g'})}{Pe_g'} \right] \right\}$$

THE PROBLEM AND ITS SOLUTION

Let us consider a porous plate (Fig. 1). Along one of its surfaces a hot gas stream is flowing. From the other surface coolant fluid is entering in a direction opposite to that of the conduction of heat in the porous plate. Heat is absorbed by the coolant fluid from the porous wall and its temperature rises. Heat balance of an element unit of area and thickness dy at a distance y from interface yields the following differential equation.

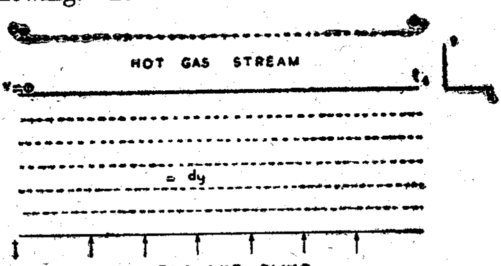

Fig. 1—Porous plate vis-a-vis hot gas and coolant fluid.

$$\frac{d^3t}{dy^3} + \frac{h'}{GC_p}\frac{d^2t}{dy^2} - \frac{h'}{K_s}\frac{dt}{dy} = 0 \quad (1)$$

To find out the temperature distribution in hot gas stream we assume that the thermal and physical characteristics of the gas are not disturbed by the introduction of coolant fluid. We also assume the flow to be a simple Coutte flow with the modification that there is a velocity v along y axis caused by the flow of coolant in the stream.

From the momentum equation

$$u \frac{du}{dx} + v \frac{du}{dy} = \frac{\mu}{\rho} \frac{d^2u}{dy^2}$$

we have, therefore, in this

$$\frac{du}{dy} = \frac{\mu}{G} \frac{d^2u}{dy^2} \tag{2}$$

since $\rho v = G$ and $\frac{du}{dx} = 0$

Boundary conditions are

$$U = U_\infty \quad \text{at} \quad y = \eta$$
$$U = 0 \quad \text{at} \quad y = 0$$

The solution of this equation is

$$U = \frac{U_\infty}{1 - e^{\frac{G}{\mu}\eta}} \left(1 - e^{\frac{G}{\mu}y}\right)$$

Putting the above result in the energy equation

$$U \frac{d\theta}{dx} + v \frac{d\theta}{dy} = \frac{Kg}{\rho C_p} \frac{d^2\theta}{dy^2} + \frac{\mu}{\rho C_p} \left(\frac{du}{dy}\right)^2$$

we have

$$\frac{K_g}{GC_p} \frac{d^2\theta}{dy^2} - \frac{d\theta}{dy} + \frac{\mu}{GC_p} \left[- \frac{G/\mu \; e^{\frac{G}{\mu}y} \; U_\infty}{1 - e^{\frac{G}{\mu}\eta}} \right]^2 = 0 \tag{3}$$

as in our case $\frac{d\theta}{dx} = 0$

Equations (1) and (3) can be non-dimensionalised by the following substitutions :

$$v_1 = \frac{t - t_0}{\theta_\infty - t_0} \qquad \xi_1 = \frac{y}{\delta}$$

$$v_2 = \frac{\theta - t_0}{\theta_\infty - t_0} \qquad \xi_2 = \frac{y}{\eta}$$

The transformed equations will be

$$\frac{d^3v_1}{d\xi_1^3} + \frac{h'\delta}{GC_p} \frac{d^2v_1}{d\xi_1^2} - \frac{h'\delta^2}{K_s} \frac{dv_1}{d\xi_1} = 0 \tag{4}$$

and

$$\frac{d^2v_2}{d\xi_2^2} - \frac{GC_p\,\eta}{K_g}\frac{dv_2}{d\xi_2} + \frac{G^2\eta^2 U_\infty^{\,2}}{\mu\, K_g(\theta_\infty - t_0)}\;\frac{e^{\frac{2G\eta}{\mu}\xi_2}}{\left(1 - e^{\frac{G}{\mu}\eta}\right)^2} = 0 \tag{5}$$

or

$$\frac{d^3v_1}{d\xi_1^3} + St_c'\frac{d^2v_1}{d\xi_1^2} - K\,Nu_c'\frac{dv_1}{d\xi_1} = 0 \tag{6}$$

and

$$\frac{d^2v_2}{d\xi_2^2} - Pe_g'\frac{dv_2}{d\xi_2} + Pr_g\,E_g'\,Re_g'^2\frac{e^{2Re_g'\xi_2}}{(1 - e^{Re_g'})^2} = 0 \tag{7}$$

Solution of equations (6) and (7) is

$$v_1 = \alpha + \beta\, e^{S\xi_1} + \gamma\, e^{-R\xi_1} \tag{8}$$

and

$$v_2 = \alpha' + \beta'\, e^{Pe_g'\xi_2} + \frac{Pr_g\,E_g'}{(1 - e^{Re_g'})^2}\;\frac{e^{2Re_g'\xi_2}}{2\,(Pr_g - 2)} \tag{9}$$

Boundary and other conditions at the contact surface are :

at $\quad y = 0,\ \ \theta = t$

and $\quad K_g\dfrac{d\theta}{dy} = K_s\dfrac{dt}{dy}$

also $\dfrac{d^2t}{dy^2} = 0$ (since we assume $t = T$ at the contact surface[5])

at $\quad y = \eta,\ \ \theta = \theta_\infty$

and

at $\quad y = -\delta,\ \ t = t_0$

The boundary conditions in the transformed form are :

at $\quad \xi_1 = \xi_2 = 0$

$$v_1 = v_2 \tag{10}$$

$$\frac{dv_1}{d\xi_1} = \lambda K\frac{dv_2}{d\xi_2} \tag{11}$$

and

$$\frac{d^2v_1}{d\xi_1^2} = 0 \tag{12}$$

at

$$\xi_1 = -1,\quad v_1 = 0 \tag{13}$$

and

at

$$\xi_2 = 1,\quad v_2 = 1 \tag{14}$$

Eliminating α, β, γ, α' and β' in (8) and (9) with the help of equations (10)—(14), we have

$$v_1 = \frac{\pi}{\triangle}\left\{S^2\left(e^{-R\xi_1} - e^{R}\right) - R^2\left(e^{S\xi_1} - e^{-S}\right)\right\} \tag{15}$$

and $$v_2 = \frac{Pr_g\, E_g'\, e^{2Re\xi_2}}{2\,(Pr_g - 2)\,(1 - e^{Re_g'})^2} + 1 - \frac{Pr_g\, E_g'\, e^{2Re_g'}}{2\,(Pr_g - 2)\,(1 - e^{Re_g'})^2}$$

$$- \left[\frac{(R + S)\, RS}{\lambda\, K\, Pe_g'} \, \frac{\pi}{\triangle} + \frac{E_g'}{(Pr_g - 2)\,(1 - e^{Re_g'})^2} \right] (e^{Pe_g'\xi_2} - e^{Pe_g'}) \quad (16)$$

When there is no injection, velocity profile for hot gas stream reduces to

$$u = \frac{U_\infty}{\eta}\, y$$

and in that case equations for temperature distributions in porous body and hot gas stream are:

$$\frac{d^2 v_1}{d\xi_1^2} = 0$$

and $$\frac{d^2 v_2}{d\xi_2^2} = - Pr_g\, E_g'$$

with the following boundary conditions :

at $$\xi_1 = \xi_2 = 0$$

$$v_1 = v_2$$

and $$\frac{dv_1}{d\xi_1} = \lambda K \frac{dv_2}{d\xi_2}$$

at $$\xi_1 = -1\,, \quad v_1 = 0$$

and

at $$\xi_2 = 1\,, \quad v_2 = 1$$

This system of equations yields the following solutions :

$$v_1 = \frac{\lambda\, K\,(2 + Pr_g\, E_g')}{2\,(1 + \lambda K)}\,(1 + \xi_1) \quad (17)$$

and $$v_2 = \frac{1}{1 + \lambda\, K} \left(1 + \frac{Pr_g\, E_g'}{2}\right) \xi_2 - \frac{Pr_g\, E_g'}{2}\, \xi_2^2$$

$$+ \left(\frac{\lambda K}{1 + \lambda\, K}\right) \times \left(1 + \frac{Pr_g\, E_g'}{2}\right) \quad (18)$$

RESULTS AND DISCUSSION

Numerical values of v_1 and v_2 with and without injection have been exhibited at Fig. 2—6 for various values of conductivity ratio K and Eckert number. The values of other non-dimensional numbers have been taken as

$$Nu_c' = 2 \qquad \lambda = 1$$
$$St_c' = 5 \qquad E_g' = 0,\ 1 \text{ and } 4$$
$$Pr_g = 0{\cdot}5 \qquad K = 0,\ 0{\cdot}2,\ 0{\cdot}5 \text{ and } 1$$
$$Re_g' = 5$$

In every graph x-axis represents the non-dimensional thickness of the porous body from $\xi_1 = -1$ to $\xi_1 = 0$ and non-dimensional thickness of hot gas layer from $\xi_2 = 0$ to $\xi_2 = 1$. $\xi_1 = \xi_2 = 0$ is the contact surface. v stands for the non-dimensional temperature with n the porous body as well as gas layer (i.e. for v_1 and v_2) and is represented along the ordinate.

We have considered two cases one with injection and the other without injection. The examination of the curves at Fig. 2—7 reveals that the temperature distribution in porous plate is not linear when there is injection. Also the contact surface temperature is reduced.

In the absence of injection (Fig. 1—3) temperature gradient at the contact surface in the porous body differs sharply from the temperature gradient at the contact surface in the hot gas layer. The injection (Fig. 5—7) somewhat smoothens this difference which shows that the coolant cools not only the porous plate but also the lower layers of the hot gas stream.

Increase in Eckert number causes an increase in contact surface temperature and thereby the temperature distribution in porous plate and hot gas stream are modified. When there is no injection, we see that as Eckert number increases, the curves showing temperature distribution in hot gas layer (Fig. 2—4) becomes more and more convex upwards which shows that flux increases constantly as we move towards contact surface. This is expected because of the heat generated in the hot gas layer due to friction.

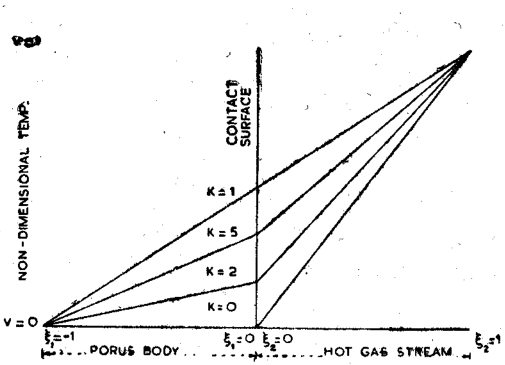

Fig. 2—Temperature distribution for different conductivity ratio K when Eckert number $E_g = 0$ and injection is absent.

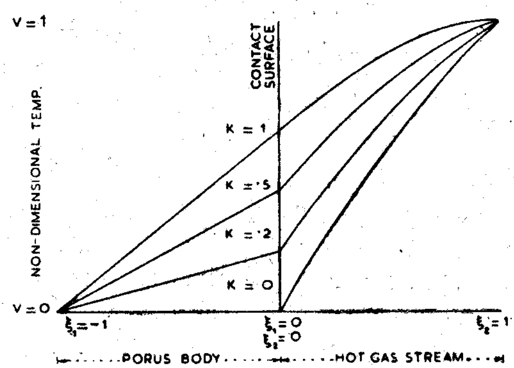

Fig. 3—Temperature distribution for different conductivity ratio K when Eckert number $E_g = 1$ and injection is absent.

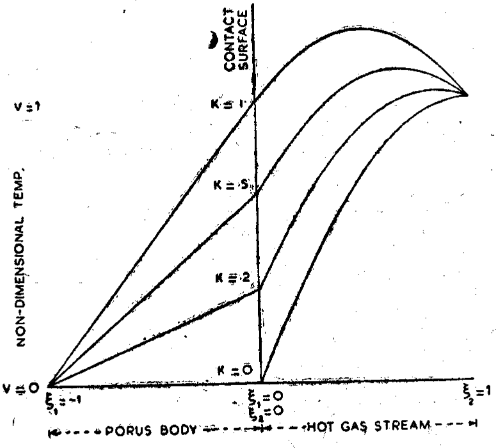

Fig. 4—Temperature distribution for different conductivity ratio K when Eckert number $E_g = 4$ and injection is absent.

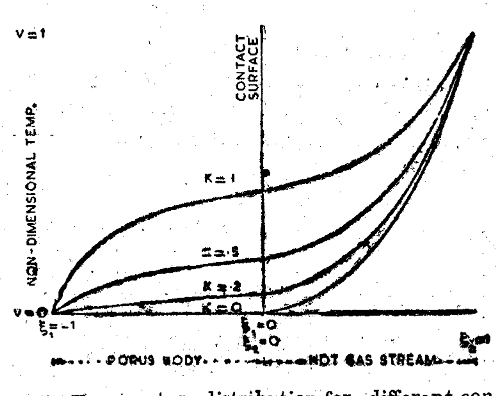

Fig. 5—Temperature distribution for different conductivity ratio K when Eckert number $E_g = 0$ and injection is present.

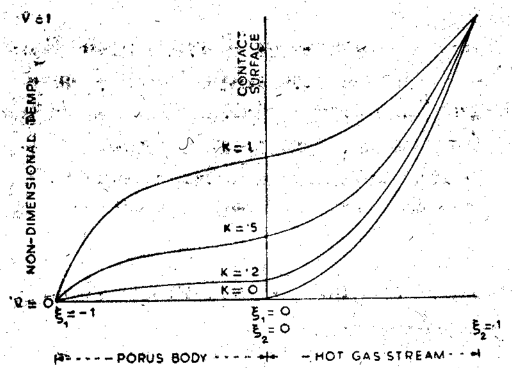

Fig. 6—Temperature distribution for different conductivity ratio K when Eckert number $E_g = 1$ and injection is present.

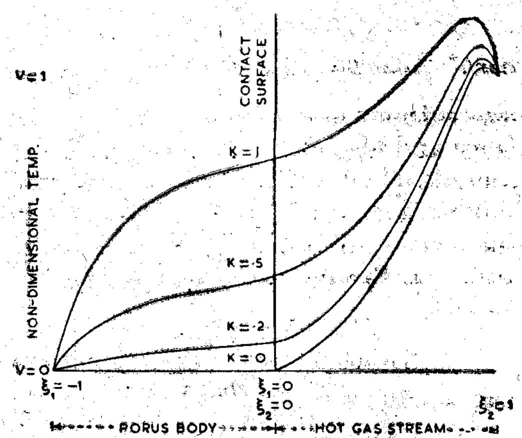

Fig. 7—Temperature distribution for different conductivity ratio K when Eckert number $E_g = 4$ and injection is present.

In case of injection we see that these curves (Fig. 5—7) are concave near the contact surface which shows that the flux goes on decreasing towards contact surface. This effect is due to the injection of coolant which absorbs much heat. However, very near the top layer, curves are again convex specially when Eckert number is larger (Fig. 7). This is to be expected as the heat generation in the top layers is more prominent as compared to the absorption of the heat by the coolant.

Parameter K represents the effect of conductivity of the solid surface on heat transfer in hot gas. A study of the graphs shows that as the conductivity of plate increases (i. e. K decreases) contact surface temperature reduces affecting the temperature distribution in hot gas. This effect is more marked when Eckert number is high and injection is absent.

If we compare graphs for $K = 1$ in Fig. 2—4 with those in Fig. 5—7 it will be observed that though the injection brings down the contact surface temperature, the temperature at some of the inner points of the porous body becomes higher due to injection. The reason for this fallacy is that this value of $K = 1$ is unrealistic (conductivities of gases are much lower than those of solids) and violates certain assumptions made in the analysis of the problem.

ACKNOWLEDGEMENTS

The author is very much grateful to Dr. I. J. Kumar, for suggesting the problem and giving many helpful suggestions. He is also thankful to Dr. R. R. Aggarwal, Assistant Director, Mathematics for encouragement throughout the preparartion of this paper. Thanks are also due to the Director, D.S.L. for permission to publish it.

REFERENCES

1. GROOTENHUIS, P. & MOORE N. P. W., *Proc. VII Internat. Cong. Appl. Mech.*, London, **3** (1948), 106.
2. WEINBAUM, S. & WHEELER H. L., *J. Appl. Phys.*, **20** (1949), 113.
3. GREEN LEON JR. & CALF DOWNEY, *J. Appl. Mech.* **19** (1952), 173.
4. MAYER, E. & BARTAS J. G., *Jet Propulsion*, **24** (1954), 366.
5. GROOTENHUIS, P., *J. Roy. Aeronautical Soc.* **63** Feb. (1959), 73.
6. BLAND, D. R., *Proc. Roy. Soc.*, **A 221**, 1144 (1954), 1.
7. SCHLICHTING, H., "Boundary Layer Theory", Translated by DR. J. KESTIN, (Mc-Graw Hill Book Co. Inc. 330 West 42nd Street, New York 36, N. Y., (U. S. A.) 1960.
8. FRIEDMAN, J., *J. Amer. Rocket Soc.* No. 79, Dec. (1949), 147.
9. PERELMAN T. L., "Heat and Mass Transfer", Vol. 5, In Russian, (U. S. S. R. Academy of Sciences, Minsk) 1963, p. 79.