UNSTEADY FLOW OF BINGHAM PLASTIC BETWEEN TWO FIXED COAXIAL CYLINDERS UNDER TIME DEPENDENT PRESSURE GRADIENT

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The unsteady flow of Bingham plastic between two fixed coaxial cylinders under time dependent pressure gradient has been discussed. It is found that the flow is possible in the whole of the region for the small interval of time. The solution is obtained in terms of Bessel function and the results are presented graphically.

We consider the unsteady flow of the material which can support a finite stress elastically without flow and which flows with a constant plastic fluidity (mobility) when the stresses are sufficiently great. Following Bingham[1] and Houwink[2] such a material is called "Bingham plastic". Oldroyd[3] has formulated the constitutive equations for such a material. The problems of unsteady rectilinear plastic flow between parallel planes and through a pipe of circular section were studied by Oldroyd[4]. Recently Paria[5] investigated the rotatory flow, both steady and unsteady, between coaxial circular cylinders of moving boundaries. We have[6] studied the unsteady flow of Bingham plastic between two eccentric circular cylinders and confocal elliptic cylinders.

In this paper the unsteady flow of Bingham plastic between two fixed coaxial cylinders under time dependent pressure gradient has been studied. We consider the case when whole region is in motion. It is found that such a motion is possible for a small interval of time and as the time increases to infinity the velocity becomes constant.

STATEMENT OF THE PROBLEM AND BOUNDARY CONDITIONS

We consider the unsteady flow of Bingham plastic between two fixed coaxial cylinders of radii a and b $(b > a)$. Let us assume that initially the material is at rest and the flow is caused under the influence of a pressure gradient which is a suitable function of time. We select the cylindrical coordinates (r, θ, z), z-axis coinciding with the common axis of the cylinders. Let u_r, u_θ and u_z be the components of the velocity in the direction of r, θ and z respectively, then by symmetry $u_r = 0$, $u_\theta = 0$ while u_z is independent of θ and z but function of r and t, i.e. $u_z = u\,(r, t)$. Let the pressure gradient be

$$\frac{1}{\rho}\,\frac{\partial p}{\partial r} = P\,(t) \tag{1}$$

The boundary conditions are

$$\left.\begin{array}{ll} u = 0 & r = a \\ u = 0 & r = b \end{array}\right\} \quad t > 0 \tag{2}$$

and the initial conditions are

$$u = 0 \text{ at } t = 0$$

FUNDAMENTAL EQUATION

The rheological equations of state[3] are :

Elastic Region

$$\left.\begin{array}{l} p_{ii} = 3\kappa\Delta \\ p'_{ik} = 2\mu e'_{ik} \end{array}\right\} \quad \left(\frac{1}{2}\,p'_{ik}\,p'_{ik} \leqslant Y^2\right) \tag{3}$$

Flow Region

$$p'_{ik} = 2\eta\, e'_{ik} \left(\frac{1}{2} p'_{ik}\, p'_{ik} \geqslant Y^2\right) \tag{4}$$

with
$$\eta = \eta_1 + Y\, (2e'_{ik}\, e'_{ik})^{-\frac{1}{2}}$$

where η_1 is the (constant) reciprocal mobility, Y is the (constant) yield value, μ is the (constant) rigidity modulus in the elastic region and κ (not necessarily constant) is the bulk modulus, whereas ϵ_{ij} is the strain tensor, e_{ij} is the rate of strain tensor, $\Delta = \epsilon_{ii}$ is the delatation, p'_{ik} is the stress tensor. The primes denote the deviatoric components of tensor, i.e.

$$p'_{ik} = p_{ik} + p\delta_{ik}\,, \quad p = -\frac{1}{3}\, p_{ii} \tag{5}$$

δ_{ik} being substitution tensor.

The elastic region is treated as rigid and the transition conditions to be satisfied on the yield surface are

(i) The velocity must be continuous.

(ii) e'_{ik} must vanish identically.

The equations of motion and continuity are

$$\rho\, \frac{D\omega_i}{Dt} = \frac{\partial p'_{ik}}{\partial x_k} - \frac{\partial p}{\partial x_i} + \rho X_i \tag{6}$$

and

$$\frac{D\rho}{Dt} + \rho e_{ii} = 0 \tag{7}$$

where ω_i is the velocity vector, X_i is the external force vector, ρ is the density and $\frac{D}{Dt}$ denotes the differentiation with respect to time following the material particle. We also have

$$e_{ij} = \frac{1}{2}\, (\omega_{i,j} + \omega_{j,i}) \tag{8}$$

The above equations with appropriate boundary conditions on stresses, velocity and pressure determine the velocity field.

SOLUTION OF THE PROBLEM

We assume the velocity components in the cylindrical coordinates (r, θ, z) to be

$$u_r = 0\,, \quad u_\theta = 0\,, \quad u_z = u\,(r\,,\, t)$$

where u is the function of r and t and pressure gradient $\frac{1}{\rho}\, \frac{\partial p}{\partial z} = P\,(t)$. The non-vanishing component of rate of strain tensor is found to be

$$e'_{rz} = e_{rz} = \frac{1}{2}\, \frac{\partial u}{\partial r} \tag{9}$$

The corresponding component of stress tensor is given by

$$p'_{rz} = p_{rz} = 2\eta e_{rz} + \frac{Y e_{rz}}{|\, e_{rz}\, |} \tag{10}$$

if e_{rz} is assumed to be positive every where in the flow region,

$$p'_{rz} = p_{rz} = 2\eta e_{rz} + Y \tag{11}$$

The equation of continuity is satisfied identically if ρ is taken as constant while in absence of body forces the equation of motion reduces to

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial z} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r p_{rz})\right] \tag{12}$$

Let us take $\quad -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} = P(t) = Qe^{-\alpha t}$ (say) $(\alpha > 0)$

where Q and α are constants.

Then equation (12) becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{\nu}\frac{\partial u}{\partial t} = \frac{P(t)}{\nu} - \frac{Y}{\eta_1}\frac{1}{r} \tag{13}$$

where

$$\nu = \eta_1/\rho$$

The Laplace transform $\bar{u}(r, \zeta) = \int_0^\infty u(r, t)\exp(-\zeta t)\,dt$ of (13) and (2) reduces the problem to the solution of the differential equation

$$\frac{d^2 \bar{u}}{dr^2} + \frac{1}{r}\frac{d\bar{u}}{dr} - q^2\bar{u} = \frac{\bar{P}(\zeta)}{\nu} - \frac{Y}{\eta_1 \zeta}\frac{1}{r} \tag{14}$$

where

$$q^2 = \zeta/\nu$$

with the boundary conditions

$$\left.\begin{aligned} \bar{u} &= 0 \ \text{ for } r = a \ \text{(internal)} \\ \bar{u} &= 0 \ \text{ for } r = b \ \text{(external)} \end{aligned}\right\} \tag{15}$$

and $\quad \bar{P}(\zeta) = \int_0^\infty e^{-\zeta t} P(t)\,dt = Q\int_0^\infty e^{-\zeta t} e^{-\alpha t}\,dt = \dfrac{Q}{\zeta + \alpha}$

The solution of (14) can be taken[7] as

$$\bar{u} = C_1 \mathbf{I}_0(qr) + C_2 \mathbf{K}_0(qr) - \frac{\bar{P}(\zeta)}{\zeta} - \frac{\pi Y}{2\sqrt{\eta_1 \rho}}\zeta^{-3/2}\mathbf{L}_0(qr) \tag{16}$$

where $\mathbf{I}_0(qr)$ and $\mathbf{K}_0(qr)$ are the modified Bessel functions of first and second kind respectively of zero order and $\mathbf{L}_0(qr)$ is the modified Struve function of zeroth order.

Now using the boundary conditions (15) we have

$$C_1 = \frac{\bar{P}(\zeta)}{\zeta}\left[\frac{\mathbf{K}_0(qb) - \mathbf{K}_0(qa)}{\mathbf{I}_0(qa)\mathbf{K}_0(qb) - \mathbf{I}_0(qb)\mathbf{K}_0(qa)}\right] - \frac{\pi Y}{2\sqrt{\eta_1 \rho}}\zeta^{-3/2}\left[\frac{\mathbf{L}_0(qb)\mathbf{L}_0(qa) - \mathbf{L}_0(qa)\mathbf{K}_0(qb)}{\mathbf{I}_0(qa)\mathbf{K}_0(qb) - \mathbf{I}_0(qb)\mathbf{K}_0(qa)}\right]$$

$$C_2 = \frac{\bar{P}(\zeta)}{\zeta}\left[\frac{\mathbf{I}_0(qa) - \mathbf{I}_0(qb)}{\mathbf{I}_0(qa)\mathbf{K}_0(qb) - \mathbf{I}_0(qb)\mathbf{K}_0(qa)}\right] + \frac{\pi Y}{2\sqrt{\eta_1 \rho}}\zeta^{-3/2}\left[\frac{\mathbf{L}_0(qb)\mathbf{I}_0(qa) - \mathbf{L}_0(qa)\mathbf{I}_0(qb)}{\mathbf{I}_0(qa)\mathbf{K}_0(qb) - \mathbf{I}_0(qb)\mathbf{K}_0(qa)}\right]$$

so that

$$\bar{u} = \frac{\bar{P}(\zeta)}{\zeta}\left[\frac{\{\mathbf{I}_0(qa)-\mathbf{I}_0(qb)\}\,\mathbf{K}_0(qr)-\{\mathbf{K}_0(qa)-\mathbf{K}_0(qb)\}\,\mathbf{I}_0(qr)}{\mathbf{I}_0(qa)\,\mathbf{K}_0(qb)-\mathbf{I}_0(qb)\,\mathbf{K}_0(qa)}-1\right]+$$

$$+\frac{\pi Y}{2\sqrt{\eta_1\rho}}\,\zeta^{-3/2}\left[\frac{\{\mathbf{L}_0(qa)\,\mathbf{K}_0(qb)-\mathbf{L}_0(qb)\,\mathbf{K}_0(qa)\}\,\mathbf{I}_0(qr)}{\mathbf{I}_0(qa)\,\mathbf{K}_0(qb)-\mathbf{I}_0(qb)\,\mathbf{K}_0(qa)}-\right.$$

$$\left.-\frac{\{\mathbf{L}_0(qa)\,\mathbf{I}_0(qb)-\mathbf{L}_0(qb)\,\mathbf{I}_0(qa)\}\,\mathbf{K}_0(qr)}{\mathbf{I}_0(qa)\,\mathbf{K}_0(qb)-\mathbf{I}_0(qb)\,\mathbf{K}_0(qa)}-\mathbf{L}_0(qr)\right] \tag{17}$$

Hence by the theorem of Inversion[8]

$$u=\frac{Q}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}}{\lambda(\lambda+\alpha)}\times$$

$$\left[\frac{\mathbf{K}_0\{r(\lambda/\nu)^{\frac12}\}[\mathbf{I}_0\{a(\lambda/\nu)^{\frac12}\}-\mathbf{I}_0\{b(\lambda/\nu)^{\frac12}\}]-\mathbf{I}_0\{r(\lambda/\nu)^{\frac12}\}[\mathbf{K}_0\{a(\lambda/\nu)^{\frac12}\}-\mathbf{K}_0\{b(\lambda/\nu)\}^{\frac12}]}{\mathbf{I}_0\{a(\lambda/\nu)^{\frac12}\}\,\mathbf{K}_0\{b(\lambda/\nu)^{\frac12}\}-\mathbf{I}_0\{b(\lambda/\nu)^{\frac12}\}\quad\mathbf{K}_0\{a(\lambda/\nu)^{\frac12}\}}-1\right]d\lambda$$

$$+\frac{1}{2\pi i}\cdot\frac{\pi Y}{2(\eta_1\rho)^{\frac12}}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}}{\lambda^{3/2}}\cdot$$

$$\left[\frac{\mathbf{I}_0\{r(\lambda/\nu)^{\frac12}\}[\mathbf{L}_0\{a(\lambda/\nu)^{\frac12}\}\mathbf{K}_0\{b(\lambda/\nu)^{\frac12}\}-\mathbf{L}_0\{b(\lambda/\nu)^{\frac12}\}\mathbf{K}_0\{a(\lambda/\nu)^{\frac12}\}]}{\mathbf{I}_0\{a(\lambda/\nu)^{\frac12}\}\,.\,\mathbf{K}_0\{b(\lambda/\nu)^{\frac12}\}-\mathbf{I}_0\{b(\lambda/\nu)^{\frac12}\}\mathbf{K}_0\{a(\lambda/\nu)^{\frac12}\}}\right.$$

$$-\frac{\mathbf{K}_0\{r(\lambda/\nu)^{\frac12}\}\;[\mathbf{L}_0\{a(\lambda/\nu)^{\frac12}\}\,\mathbf{I}_0\{b(\lambda/\nu)^{\frac12}\}-\mathbf{L}_0\{b(\lambda/\nu)^{\frac12}\}.\;\mathbf{I}_0\{a(\lambda/\nu)^{\frac12}\}]}{\mathbf{I}_0\{a(\lambda/\nu)^{\frac12}\}\,\mathbf{K}_0\{b(\lambda/\nu)^{\frac12}\}-\mathbf{I}_0\{b(\lambda/\nu)^{\frac12}\}\;\mathbf{K}_0\{a(\lambda/\nu)^{\frac12}\}}$$

$$\left.-\;\mathbf{L}_0\{r(\lambda/\nu)^{\frac12}\}\right]d\lambda \tag{18}$$

EVALUATION OF THE COMPLEX INTEGRAL

γ is choosen such that it is greater than the real part of all singularities of the integrand in (18). In the first part of (18), the integrand is a single valued function of λ and has pole at $\lambda = 0$ and $\lambda = -\alpha$ (a pole of order one) and the other poles are the zeros of the denominator. To find these zeros put $\lambda = -\nu\beta^2$, then it becomes

$$\mathbf{I}_0(ia\beta)\,\mathbf{K}_0(ib\beta)-\mathbf{I}_0(ib\beta)\,\mathbf{K}_0\;(ib\beta)\;=-\frac{\pi}{2}\left\{\mathbf{J}_0(a\beta)\,\mathbf{Y}_0(b\beta)-\mathbf{J}_0(b\beta)\,\mathbf{Y}_0(a\beta)\right\}$$

Let β_1, β_2 ...... β_s be the roots of the equation

$$\mathbf{J}_0(a\beta)\,\mathbf{Y}_0(b\beta) - \mathbf{J}_0(b\beta)\,\mathbf{Y}_0(a\beta) = 0 \tag{19}$$

Then the zeros are

$$\lambda = -\nu\beta_1^2,\ -\nu\beta_2^2,\ \ldots\ldots,\ -\nu\beta_s^2$$

These are the simple poles of the integrand.

Hence by Cauchy's theorem of residue, we obtain

$$u = \frac{Qe^{-\alpha t}}{\alpha}\cdot$$

$$\left[1+\frac{\mathbf{J}_0\{r(\alpha/\nu)^{\frac12}\}[\mathbf{Y}_0\{a(\alpha/\nu)^{\frac12}\}-\mathbf{Y}_0\{b(\alpha/\nu)^{\frac12}\}]-\mathbf{Y}_0\{r(\alpha/\nu)^{\frac12}\}[\mathbf{J}_0\{a(\alpha/\nu)^{\frac12}\}-\mathbf{J}_0\{b(\alpha/\nu)^{\frac12}\}]}{\mathbf{J}_0\{a(\alpha/\nu)^{\frac12}\}\,\mathbf{Y}_0\{b(\alpha/\nu)^{\frac12}\}-\mathbf{J}_0\{b(\alpha/\nu)^{\frac12}\}\,\mathbf{Y}_0\{a(\alpha/\nu)^{\frac12}\}}+\right.$$

$$+\ \pi Q\sum_{s=1}^{\infty}\frac{e^{-\nu\beta_s^2 t}}{\alpha-\nu\beta_s^2}\ \frac{\mathbf{J}_0(a\beta_s)\,\mathbf{J}_0(b\beta_s)}{\mathbf{J}_0^2(a\beta_s)-\mathbf{J}_0^2(b\beta_s)}\left[\mathbf{J}_0(r\beta_s)\left\{\mathbf{Y}_0(b\beta_s)-\mathbf{Y}_0(a\beta_s)\right\}-\right.$$

$$\left.\left.-\mathbf{Y}_0(r\beta_s)\left\{\mathbf{J}_0(b\beta_s)-\mathbf{J}_0(a\beta_s)\right\}\right]\right]+\frac{\pi^2}{2}\frac{Y}{\eta_1}\sum_{s=1}^{\infty}\frac{e^{-\nu\beta_s^2 t}}{\beta_s}\cdot$$

$$\cdot\frac{\mathbf{J}_0(a\beta_s)\,\mathbf{J}_0(b\beta_s)}{\mathbf{J}_0^2(a\beta_s)-\mathbf{J}_0^2(b\beta_s)}\left[\mathbf{J}_0(r\beta_s)\left\{\mathbf{H}_0(a\beta_s)\,\mathbf{Y}_0(b\beta_s)-\mathbf{H}_0(b\beta_s)\,\mathbf{Y}_0(a\beta_s)\right\}-\right.$$

$$\left.-\mathbf{Y}_0(r\beta_s)\left\{\mathbf{H}_0(a\beta_s)\,\mathbf{J}_0(b\beta_s)-\mathbf{H}_0(b\beta_s)\,\mathbf{J}_0(a\beta_s)\right\}\right]+$$

$$+\ \frac{Y}{\eta_1}\left\{\frac{a\log\dfrac{b}{r}+b\log\dfrac{r}{a}}{\log\dfrac{b}{a}}-r\right\} \tag{20}$$

NUMERICAL VALUES

For the simplification of (20) we choose $Q=\dfrac{Y}{a\rho}$. Also we put

$$a\beta_s=\phi_s,\quad \nu t/a^2=\tau,\quad \alpha a^2/\nu=\psi \tag{21}$$

To introduce these dimensionless quantities (21), (20) may be written as

$$\frac{u}{Ya/\eta_1}=\frac{e^{-\tau}}{\psi}\cdot$$

$$\cdot\left\{1+\frac{\mathbf{J}_0\left(\frac{r}{a}\sqrt{\psi}\right)\left\{\mathbf{Y}_0\sqrt{\psi}-\mathbf{Y}_0\left(\frac{b}{a}\sqrt{\psi}\right)\right\}-\mathbf{Y}_0\left(\frac{r}{a}\sqrt{\psi}\right)\left\{\mathbf{J}_0\sqrt{\psi}-\mathbf{J}_0\left(\frac{b}{a}\sqrt{\psi}\right)\right\}}{\mathbf{J}_0\left(\sqrt{\psi}\right)\mathbf{Y}_0\left(\frac{b}{a}\sqrt{\psi}\right)-\mathbf{J}_0\left(\frac{b}{a}\sqrt{\psi}\right)\mathbf{Y}_0\sqrt{\psi}}\right\}+$$

$$+\ \frac{\pi}{\psi}\sum_{s=1}^{\infty}\frac{e^{-\phi_s^{2}\tau}}{1-\frac{\phi_s^{2}}{\psi}}\ \frac{\mathbf{J}_0(\phi_s)\,\mathbf{J}_0\left(\frac{b}{a}\phi_s\right)}{\mathbf{J}_0^2(\phi_s)-\mathbf{J}_0^2\left(\frac{b}{a}\phi_s\right)}\left[\mathbf{J}_0\left(\frac{r}{a}\phi_s\right)\left\{\mathbf{Y}_0\left(\frac{b}{a}\phi_s\right)-\mathbf{Y}_0(\phi_s)\right\}-\right.$$

$$\left.-\ \mathbf{Y}_0\left(\frac{r}{a}\phi_s\right)\left\{\mathbf{J}_0\left(\frac{b}{a}\phi_s\right)-\mathbf{J}_0(\phi_s)\right\}\right]+\frac{\pi^2}{2}\sum_{s=1}^{\infty}\frac{e^{-\phi_s^{2}\tau}}{\phi_s}\ \frac{\mathbf{J}_0(\phi_s)\,\mathbf{J}_0\left(\frac{b}{a}\phi_s\right)}{\mathbf{J}_0^2(\phi_s)-\mathbf{J}_0^2\left(\frac{b}{a}\phi_s\right)}$$

$$\left[\mathbf{J}_0\left(\frac{r}{a}\phi_s\right)\left\{\mathbf{H}_0(\phi_s)\,\mathbf{Y}_0\left(\frac{b}{a}\phi_s\right)\mathbf{H}_0\left(\frac{b}{a}\phi_s\right)\mathbf{Y}_0(\phi_s)\right\}-\mathbf{Y}_0\left(\frac{r}{a}\phi_s\right)\cdot\right.$$

$$\left.\left\{\mathbf{H}_0(\phi_s)\,\mathbf{J}_0\left(\frac{b}{a}\phi_s\right)-\mathbf{H}_0\left(\frac{b}{a}\phi_s\right)\mathbf{J}_0(\phi_s)\right\}\right]+\left[\frac{\log\frac{b}{r}+\frac{b}{a}\log\frac{r}{a}}{\log\frac{b}{a}}-\frac{r}{a}\right] \qquad (22)$$

In Fig 1, $\frac{u}{\gamma a/\eta_1}$ has been plotted against $\frac{r}{a}$ for $\tau = \cdot 02$, $\psi = 1$, $\frac{b}{a} = 2$ and ϕ_s, $s = 1, 2, \ldots\ldots$ are the roots of

$$\mathbf{J}_0(\phi)\,\mathbf{Y}_0\left(\frac{b}{a}\phi\right)-\mathbf{J}_0\left(\frac{b}{a}\phi\right)\mathbf{Y}_0(\phi)=0$$

This shows that the flow is possible in the whole of the region towards ~~the axis and~~ the velocity decreases as the time increases and ultimately it becomes constant as t tends to infinity. The velocity is maximum between $\frac{r}{a} = 1\cdot 7$ and $\frac{r}{a} = 1\cdot 8$

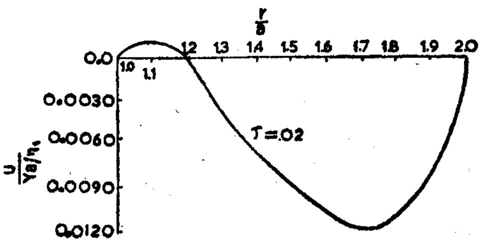

Fig. 1—Velocity distribution for different values of r/a.

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