TANGENTIAL VELOCITY PROFILE FOR AXIAL FLOW THROUGH TWO CONCENTRIC ROTATING CYLINDERS WITH RADIAL MAGNETIC FIELD

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A closed form solution of the Navier-Stokes equations has been obtained in the case of steady axisymmetric flow of an incompressible electrically conducting viscous fluid between two concentric rotating cylinders composed of an insulating material under the influence of radial magnetic field. It has been found that the velocity components are less than those of the classical hydrodynamic case. In the presence of the magnetic field, the tangential velocity becomes fully developed in a smaller axial distance than in the absence of the magnetic field. For small Reynolds number, the fully developed tangential velocity is achieved in a smaller axial distance, but it requires greater axial distance for large Reynolds number.

The flow and the heat transfer phenomena in the hydromagnetic flow between two rotating non-conducting cylinders have been studied by Ramamoorthy[1] and Jagadeesan[2]. The method of predicting the growth of a tangential velocity profile in fully developed incompressible steady laminar viscous axial flow through two coaxial cylinders when the inner surface is rotating, has been described by Astill et al[3].

In this paper, the above method is used to predict the growth of a tangential velocity profile in fully developed steady incompressible electrically conducting viscous axial fluid flow between two concentric rotating cylinders composed of an insulating material under the influence of a uniform radial magnetic field. It is assumed that the magnetic Reynolds number and Hall parameters are small and the induced magnetic field produced by the motion of the electrically conducting fluid is negligible. It is also assumed that there is no separation of charges and the surface of the body is not charged so that the electric field $\bar{E} = 0$. The axial pressure gradient is prescribed and the cylinders are rotated at speeds which are insufficient to generate Taylor vortices. The tangential velocity profile generated in a viscous fluid by the rotating cylinders in the absence of vortices becomes a function of the axial coordinate when an axial velocity is superimposed, whether or not it is fully developed[3]. However the existence of a fully developed axial velocity profile at entry to the rotating section has been confirmed by experiments[3]. In a sufficiently long annulus, the tangential profile subsequently approaches the fully developed state when it becomes independent of axial position[3].

FORMULATION OF THE PROBLEM AND FUNDAMENTAL EQUATIONS

Consider the steady flow of an incompressible electrically conducting viscous fluid between two concentric rotating non-conducting cylinders of radii a and b ($b > a$) under the influence of radial magnetic field $B_r = \dfrac{A}{r}$, where B_r is the magnetic induction in r-direction, A is a constant and r is the radial distance from the axis of the cylinder. Such a field can be produced by passing a steady current parallel to the axis of the coaxial cylinders as shown in Fig. 7·10-1 of reference 4 where the cylinders terminate at perfect electrodes which are connected through a load[4]. Another method to obtain an approximation to the desired field is by the use of a permeable core within the annulus and a permeable cylinder shell outside the annulus. The flux lines would close through these permeable paths at long distances from the region of interest. The source of the flux could be discs of permanently magnetized material between the permeable paths and the annulus channel[5]. Assume that the two cylinders, which are sufficiently long, rotate in the same direction with angular velocities ω_1 and ω_2 respectively. Following Rossow[6], it is assumed that the induced magnetic field is small and hence it can be neglected.

The Navier-Stokes equations in cylindrical coordinates for steady laminar viscous axisymmetric fully developed incompressible conducting fluid flow with radial magnetic field in the absence of body forces and induced magnetic field are[3, 4] :

$$\frac{\rho\, u_\theta^2}{r} = \frac{\partial p}{\partial r} \tag{1}$$

$$u_z \frac{\partial u_\theta}{\partial z} = \nu \left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta^2}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} \right) - \frac{\sigma u_\theta B_r^2}{\rho} \tag{2}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = \nu \left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} \right) - \frac{\sigma B_r^2 u_z}{\rho} \tag{3}$$

$$\frac{\partial}{\partial r} (r B_r) = 0 \tag{4}$$

where u_θ , u_z are tangential and axial velocity components respectively, ρ and p are density and static pressure respectively, ν is kinematic viscosity and σ is electrical conductivity.

From equation (4), B_r is proportional to $\frac{1}{r}$ and since at $r = a$, $B_r = \frac{A}{a}$, it follows[4] that $B_r = \frac{A}{r}$. Introduce dimensionless variables as follows :

$$\bar{p} = \frac{p}{\frac{1}{2}\rho\,\omega_2^2\, b^2}, \quad \bar{u}_\theta = \frac{u_\theta}{\omega_2 b}, \quad \bar{u}_z = \frac{u_z}{\omega_2 b},$$

$$R = \frac{r}{b}, \bar{z} = \frac{z}{b}, \bar{B}_r = \frac{B_r}{B_0} = \frac{1}{R}, \quad H_a^2 = \frac{\sigma B_0^2 b^2}{\mu},$$

$$R_e = \frac{\omega_2 b^2}{\nu}, \ R_e \frac{\partial \bar{p}}{\partial \bar{z}} = -\alpha \ (\alpha > 0), \ \lambda = \frac{a}{b}$$

where B_0 is the characteristic magnetic induction, H_a is the Hartmann number, μ is the viscosity, R_e is Reynolds number and $\frac{\partial \bar{p}}{\partial \bar{z}}$ is the axial pressure gradient which is supposed to be prescribed.

The equations (1) — (3) can be expressed in non-dimensional form as :

$$\frac{\partial \bar{p}}{\partial R} = \frac{1}{2}\frac{\bar{u}_\theta^2}{R} \tag{5}$$

$$R_e \bar{u}_z \frac{\partial \bar{u}_\theta}{\partial \bar{z}} - \frac{\partial^2 \bar{u}_\theta}{\partial \bar{z}^2} = \frac{\partial^2 \bar{u}_\theta}{\partial R^2} + \frac{1}{R}\frac{\partial \bar{u}_\theta}{\partial R} - (1 + H_a^2)\frac{\bar{u}_\theta}{R^2} \tag{6}$$

$$\frac{\partial^2 \bar{u}_z}{\partial R^2} + \frac{1}{R}\frac{\partial \bar{u}_z}{\partial R} - \frac{H_a^2}{R^2}\,\bar{u} = -\alpha \tag{7}$$

The boundary conditions are:

$$\left.\begin{aligned} \bar{u}_\theta(R,0) &= 0 \quad (a); & \bar{u}_\theta(1,z) &= 1 \quad (b) \\ \bar{u}_\theta(\lambda,\bar{z}) &= \frac{\omega_1 a}{\omega_2 b} = N \quad (c); & \bar{u}_\theta(R,\infty) &= \bar{u}_{\theta_1} \quad (d) \end{aligned}\right\} \qquad (8)$$

$$\bar{u}_z(1) = 0 \quad ; \qquad \bar{u}_z(\lambda) = 0 \qquad (9)$$

SOLUTION OF THE PROBLEM

For a fully developed steady viscous flow with axial pressure gradient through an infinitely long concentric rotating cylinders, the Navier-Stokes equations admit a steady solution of the form[7,8]:

$$\bar{u}_r = 0\,,\ \bar{u}_\theta = V(R)\,,\ \bar{u}_z = W(R)\,,\ \frac{\partial \bar{p}}{\partial z} = \text{constant}$$

where V and W are functions of R only. Hence the solution of equation (7), for fully developed axial flow with boundary conditions given by equation (9), can be expressed as:

$$\bar{u}_z = A R^{H_a} + B R^{-H_a} + \alpha_1 R^2 \qquad (10)$$

where

$$A = \frac{\alpha_1 (1 - \lambda^{2+H_a})}{(\lambda^{2H_a} - 1)}$$

$$B = \frac{\alpha_1 (\lambda^{2-H_a} - 1)}{(1 - \lambda^{-2H_a})}$$

$$\alpha_1 = -\frac{\alpha}{4 - H_a^2} \qquad (\alpha = \text{constant})$$

The mean axial velocity component $(\bar{u}_z)_m$, is given by:

$$(\bar{u}_z)_m = \frac{2}{1-\lambda^2}\int_\lambda^1 R\bar{u}_z\, dR$$

$$= \frac{2}{1-\lambda^2}\left[\frac{A}{2+H_a}(1-\lambda^{2+H_a}) + \frac{B}{2-H_a}(1-\lambda^{2-H_a}) + \frac{\alpha_1}{4}\left(1-\frac{\lambda^4}{4}\right)\right] \qquad (11)$$

For the fully developed tangential velocity profile, $\bar{u}_{\theta_1}$, the equation (6) reduces to:

$$\frac{\partial^2 \bar{u}_{\theta_1}}{\partial R^2} + \frac{1}{R}\frac{\partial \bar{u}_{\theta_1}}{\partial R} - \beta^2 \frac{\bar{u}_{\theta_1}}{R^2} = 0 \qquad (12)$$

where

$$\beta^2 = 1 + H_a^2$$

The boundary conditions given by equation (8) are reduced to :

$$\bar{u}_{\theta_1}(\lambda) = N ; \ \bar{u}_{\theta_1}(1) = 1 \tag{13}$$

The solution of equation (12) with boundary conditions given by equation (13) can be written as :

$$\bar{u}_{\theta_1} = C R^{\beta} + D R^{-\beta} \tag{14}$$

where

$$C = \frac{N \lambda^{\beta} - 1}{\lambda^{2\beta} - 1} \ ; \ D = \frac{1 - N \lambda^{-\beta}}{1 - \lambda^{-2\beta}}$$

Using the difference procedure of Sparrow & Lin[9], assume that $\bar{u}_{\theta} = \bar{u}\theta_1 - \bar{u}_{\theta_2}$, where $\bar{u}_{\theta_2}$ matches $\bar{u}_{\theta_1}$ at $\bar{z} = 0$, and decays to zero when $\bar{z} \rightarrow \infty$. Substituting for $\bar{u}_{\theta}$ from above in equation (6) and replacing the fully developed axial velocity u_z by its mean value $(\bar{u}_z)_m$, we get :

$$R_e (\bar{u}_z)_m \frac{\partial \bar{u}_{\theta_2}}{\partial \bar{z}} - \frac{\partial^2 \bar{u}_{\theta_2}}{\partial \bar{z}^2} = \frac{\partial^2 \bar{u}_{\theta_2}}{\partial R^2} + \frac{1}{R}\frac{\partial \bar{u}_{\theta_2}}{\partial R} - \beta^2 \frac{\bar{u}_{\theta_2}}{R^2} \tag{15}$$

which is subject to the boundary conditions :

$$\left.\begin{array}{llr} \bar{u}_{\theta_2}(R, 0) = \bar{u}_{\theta_1}(R) & & \text{(a)} \\ \bar{u}_{\theta_2}(1, \bar{z}) = 0 & & \text{(b)} \\ \bar{u}_{\theta_2}(\lambda, \bar{z}) = 0 & & \text{(c)} \\ \bar{u}_{\theta_2}(R, \infty) = 0 & & \text{(d)} \end{array}\right\} \tag{16}$$

The solution of equation (15) with the boundary conditions given by equation (16 b, c, d) can be expressed as[3]:

$$\bar{u}_{\theta_2}(R, \bar{z}) = \sum_{n=1}^{\infty} A_n \left[G_{\beta}(k_n R) \right] \exp \left[\frac{L}{2} - S_n \right] \bar{z} \tag{17}$$

where

$$G_{\beta}(k_n R) = J_{\beta}(k_n R)\, Y_{\beta}(k_n) - J_{\beta}(k_n)\, Y_{\beta}(k_n R)$$

$$L = R_e (\bar{u}_z)_m \ ; \ S_n = \left(k_n^2 + \frac{L^2}{4}\right)^{\frac{1}{2}}$$

J_{β} and Y_{β} are βth order Bessel functions of the first and second kinds respectively and A_n is constant. The characteristic equation for the determination of the separation constant k, is expressd as :

$$J_{\beta}(k)\, Y_{\beta}(\lambda k) - J_{\beta}(\lambda k)\, Y_{\beta}(k) = 0 \tag{18}$$

where k_n is the nth positive root of the above equation. The coefficient A_n is determined in such a manner that the remaining boundary condition (16a) is satisfied. From equations (14) and (17), using the boundary condition (16a) and the orthogonal properties of Bessel functions, A_n can be expressed as[10] :

$$A_n = \frac{\pi^2 k_n^2}{2} \frac{J_\beta^2 (\lambda k_n)}{J_\beta^2 (\lambda k_n) - J_\beta^2 (k_n)} \int_\lambda^1 R \bar{u}_{\theta 1} (R) \, G_\beta (k_n R) \, dR \tag{19}$$

Equation (19) can be integrated numerically. From equations (14) and (17), the tangential velocity $\bar{u}_\theta$, can be expressed as :

$$\bar{u}_\theta = C R^\beta + D R^{-\beta} - \sum_{n=1}^{\infty} A_n \left[G_\beta (k_n R) \right] \exp \left(\frac{L}{2} - S_n \right) \bar{z} \tag{20}$$

DISCUSSION OF RESULTS

For small Reynolds number or for small L, $\left(\frac{L}{2} - S_n\right) \bar{z} \rightarrow - k_n \bar{z}$, and for large Reynolds number or for large L, $\left(\frac{L}{2} - S_n\right) \bar{z} \rightarrow - k_n^2 \bar{z}/L$. k_n increases with Hartmann number H_α. Therefore for small Reynolds number, the fully developed tangential velocity is achieved in a smaller axial length, but it requires greater axial length for large Reynolds number. The axial distance in which the tangential velocity becomes fully developed decreases as Hartmann number increases. The fully developed tangential velocity $\bar{u}_{\theta_1}$ and the fully developed axial velocity $\bar{u}_z$ are given in Fig. 1. The fully developed tangential velocity $\bar{u}_{\theta 1}$ increases as N increases, but it decreases as Hartmann number H_α increases. Similarly it is evident from equation (10) and Fig. 1 that the axial velocity increases as Reynolds number or axial pressure gradient increases, but decreases as Hartmann number H_α, increases. In the absence of the magnetic field all the equations tend to classical hydrodynamic equations.

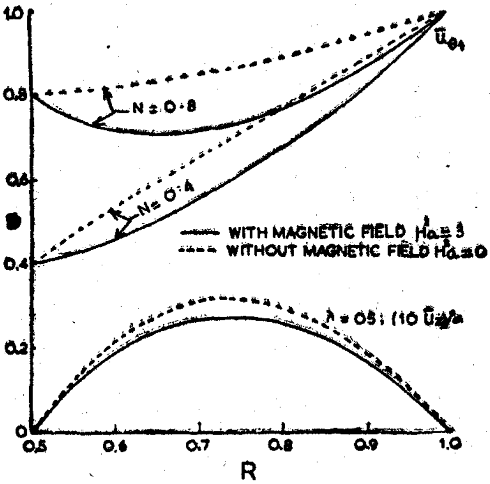

Fig. 1—*Velocity distributions.*

CONCLUSIONS

The effect of the imposition of the applied magnetic field is to decrease the velocity field. In the presence of magnetic field, the tangential velocity becomes fully developed at a smaller axial distance than in the absence of the magnetic field.

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