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Thermal Stresses Due to the Action of Heat Source

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Abstract. The propagation of thermal stresses in a half-space, due to the action of a point heat source varying in time in a harmonic manner, using the generalized theory of thermoelasticity which takes into account the effect of relaxation time is discussed.

1. Introduction

Thermal stresses arise in many familiar areas, and has been a subject of interest. They are frequently an important factor in determining material life. With the works of Biot[1], Lesson[2], Chadwick and Sneddon[3], a new trend in the research over dynamic problems of thermoelasticity has been observed since 1956, postulating a coupling between the strain field and the temperature field. Locket[4] considered the influence of the coupled strain and temperature fields on the velocity of propagation of Rayleigh surface waves. Zorski[5] concerned with stress propagation in an infinite space due to thermal impulse.

In coupled theory of thermoelasticity, if an isotropic homogeneous and elastic continuum is subjected to a mechanical or thermal disturbance, the effect of the disturbance will be felt instantaneously in both the fields as governing equations are coupled. Physically, this means that a portion of the disturbance has an infinite velocity of propagation. Such behaviour is physically inadmissible. Lord and Shulman[6] have presented the generalized theory of thermoelasticity, to eliminate this parodox by employing a more general functional relation between heat flow and temperature gradient. They considered the relaxation time as time lag needed for the onset of thermal wave.

The aim of this paper is to determine the temperature distribution and thermal stress distribution in a half space due to the action of point heat sources varying in time in a periodic manner, using the generalized theory of thermoelasticity.

2. The Governing Equations and the Solution of the Problem

The equations of the thermoelastic medium have the form

$$\mu\nabla^2 u + (\lambda + \mu)\,\text{grad div}\, u - \rho\frac{\partial^2 u}{\partial t^2} = \alpha(3\lambda + 2\mu)\,\text{grad}\,\theta \tag{1}$$

$$\nabla^2\theta \quad \frac{1}{k}\left(\frac{\partial\theta}{\partial t} + \tau_0\frac{\partial^2\theta}{\partial t^2}\right) - \eta\left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)\text{div.}\, u = -\frac{Q(p.t)}{k} \tag{2}$$

The displacement function is given by

$$u = \text{grad}\,\varphi + \text{rot}\,\psi \tag{3}$$

From Eqns. (1), (2) and (3), we get

$$\left[\nabla^{2-} - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right]\phi = \vartheta_0\theta \tag{4}$$

$$\left[\nabla^{2-} \quad \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}\right]\text{rot}\,\psi = 0 \tag{5}$$

and

$$\left[\nabla^{2-} \quad \frac{1}{k}\left(\frac{\partial}{\partial t} \quad \tau_0\frac{\partial^2}{\partial t^2}\right)\right]\theta - \eta\left[\frac{\partial}{\partial t} \quad \tau_0\frac{\partial^2}{\partial t^2}\right]\phi$$

$$\frac{Q(p,\,t)}{k} \tag{6}$$

where

K Thermal diffusivity

u = Displacement vector

θ = Temperature

α = Coefficient of linear thermal expansion

λ, μ = Lame's constants

ρ = Constant mass density

$c_1 = [(\lambda + 2\mu)/\rho]^{1/2}$ = Velocity of propagation of the logitudinal elastic wave

$c_2 = [\mu/\rho]^{1/2}$ = Velocity of propagation of the transversal elastic wave

$$\eta = \frac{\alpha(3\lambda + 2\mu)}{\rho c k}$$

$\vartheta_0 = \alpha(3\lambda + 2\mu)/(\lambda + 2\mu)$

c = Specific heat, τ_0 = Relaxation time

$$\nabla^2 \quad \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \quad \frac{\partial^2}{\partial z^2}$$

Let a point heat source act at the origin. Let us assume that this heat source vary in time in a periodic manner, therefore $Q(p, t) = e^{iwt}$. $Q_0(p)$, where w is a positive real number and denotes the frequency of heat source. In the case of periodic variation of the heat source, we have

$$\theta(p, t) = e^{iwt}\,\theta^*(p),\ \phi(p, t) = e^{iwt}\,\phi^*(p)$$
$$\psi(p, t) = e^{iwt}\,\psi^*(p) \tag{7}$$

with the help of Eqn. (7), Eqns. (4), (5) and (6) become

$$(\nabla^2 + \sigma^2)\,\varphi^* = \vartheta_0\,\theta^* \tag{8}$$

$$[\nabla^2 \quad (q - \beta)]\,\theta^* - \eta^1(q - \beta)\ \nabla^2\phi^* = -\frac{Q_0(p)}{k} \tag{9}$$

and

$$[\nabla^2 + \tau^2]\ \text{rot}\ \psi^* = 0$$

where

$$\sigma^2 = \frac{w^2}{c_1^2},\ \tau^2 = \frac{w^2}{c_2^2},\ q \quad \frac{iw}{k_1},\ \eta^1 = \eta\,k,\ \beta = \frac{\tau_0 w^2}{k}$$

$Q_0(p)$ can be written as $Q_0(p) \quad Q_0\delta(R) = \dfrac{Q_0\delta(r)\,\delta(z)}{2\pi r}$

Eliminating θ^* from Eqns. (8) and (9), we get

$$[\nabla^2 - (q - \beta)]\,[\nabla^2 + \sigma^2]\,\phi^* - (q - \beta)\,\epsilon\,\nabla^2\,\phi^* = -\frac{\vartheta_0 Q_0}{k}\ \frac{\delta(r)\,\delta(z)}{2\pi r} \tag{11}$$

where $\quad \epsilon = \eta k \vartheta_0 =$ coupling constant.

we know

$$\frac{\delta(r)\,\delta(z)}{2\pi r} \qquad \frac{}{2\pi^2}\int_0^\infty\int_0^\infty \alpha\ J_0(\alpha r)\ \text{Cos}\ Yz\ d\alpha\ dY$$

Expressing the function ϕ^* by means of the Fourier-Hankel integral

$$\phi^*(r, z) \quad \int_0^\infty\int_0^\infty A(\alpha, Y)\ J_0(\alpha r)\ \text{Cos}\ Yz\ d\alpha\ dY$$

substituting Eqns. (12) and (13) in Eqn. (11), we get

$$A(\alpha, Y) = -\frac{\vartheta_0 Q_0\,\alpha}{2\pi^2 k}\ \frac{1}{F(\alpha, Y)} \tag{14}$$

where

$$F(\alpha, Y) = [(\alpha^2 + Y^2)^2 + [\sigma^2 - (q - \beta)\,(1 + \epsilon)]\,(\alpha^2 + Y^2) - (q \quad \beta)\,\sigma^2]$$
$$= (\alpha^2 + Y^2 + k_1^2)\,(\alpha^2 + Y^2 + k_2^2) \tag{15}$$

where

$$k_1^2 \quad k_2^2 = \sigma^2 \quad (q \quad \beta)(1+\epsilon)$$

$$k_1^2\, k_2^2 \qquad (q-\beta)\,\sigma^2 \tag{16}$$

Therefore

$$\phi^*(r,z) = \frac{Q_0 \vartheta_0}{2\pi^2 k} \int_0^\infty \int_0^\infty \frac{\alpha J_0(\alpha r)}{F(\alpha, Y)} \operatorname{Cos} Yz \, d\alpha \, dY \tag{17}$$

After performing integrations, we get

$$\phi^*(r,z) = \frac{Q_0 \vartheta_0}{4\pi k\, R(k_2^2 - k_1^2)} \left[e^{-k_1 R} \quad e^{-k_2 R} \right] \tag{18}$$

where $R = [r^2 + z^2]^{1/2}$.

The temperature distribution is obtained from Eqn. (8) as

$$\theta^*(r,z) = \frac{Q_0}{4\pi k\, R(k_1^2 - k_2^2)} \left[(\sigma^2 + k^2)\, e^{-k_1 R} \quad (\sigma^2 \quad k_2^2)\, e^{-k_2 R} \right] \tag{19}$$

The knowledge of function ϕ^* enables the determination of the stresses from the equations

$$\sigma_{ij} = 2\mu \left[\frac{\partial^2}{\partial x_i\, \partial x_j} \quad \delta_{ij} \nabla^2 \right] \phi + \delta_{ij}\, \rho \frac{\partial^2 \phi}{\partial t^2} \quad i,j = 1, 2, 3 \tag{20}$$

Since the heat source is acting at the origin of the coordinate system, the temperature field, the displacements and the stresses possess spherical symmetry, they depend on the distance R from the origin and on the times. In view of the spherical symmetry, we are concerned with a modified longitudinal elastic wave, so that $\psi = 0$.

In the case of spherical symmetry,

$$\sigma_{RR} = \quad e^{iwt} \left[\frac{4\mu}{R} \frac{\partial \phi^*}{\partial R} + \rho w^2 \phi^* \right] \tag{21}$$

$$\sigma_{\phi\phi} \quad \sigma_{\vartheta\vartheta} = \quad e^{iwt} \left[2\mu \left(\frac{\partial^2 \phi^*}{\partial R^2} + \frac{1}{R} \frac{\partial \phi^*}{\partial R} \right) + \rho w^2 \phi^* \right]$$

Therefore from Eqns. (18) and (21), the stresses are given by

$$\sigma_{RR} = -\frac{e^{iwt} Q_0 \vartheta_0}{\pi k (k_2^2 - k_1^2)} \left[\left\{ -\frac{\mu}{R^3}(1 + k_1 R) + \frac{\rho w^2}{4R} \right\} e^{-K_1 R} \right.$$

$$\left. \left\{ \frac{(1 + k_2 R)\mu}{R^3} + \frac{\rho w^2}{4R} \right\} e^{-K_2 R} \right] \tag{22}$$

$$\sigma_{\phi\phi} = \sigma_{\vartheta\vartheta} \qquad \frac{e^{iwt} Q_0}{4\pi k (k_2^2 - k_1^2)} \left[\left\{ \frac{2\mu r^2}{R^5} (k_1^2 R^2 + 2k_1 R \quad 2) \right. \right.$$

$$\left(\frac{2\mu z^3}{R^5} + \frac{2\mu}{R^3} \right)(1 + k_1 R) + \frac{\rho w^2}{R} \Big\} e^{-K_1 R}$$

$$\left\{ \frac{2\mu r^2}{R^5} (k_2^2 R^2 + 2k_2 R - 2) - \left(\frac{z^2}{R^5} + \frac{1}{R^3} \right) (1+k_2 R)\, 2\mu + \frac{\rho w^2}{R} \right\} e^{-K_2 R} \Big]. \tag{23}$$

3. Conclusion

The results obtained are more general, which includes the effect of relaxation time also. The results of the coupled theory can be obtained by putting $\beta = 0$ and that of uncoupled theory by putting $\epsilon = 0$.

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