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Note on MIL-STD-1235 (ORD) Continuous Sampling Procedures for Markov-Dependent Processes

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Abstract. The paper extends the procedures of single and two-level continuous sampling plans for Markov-dependent processes under a non-replacement assumption. The average outgoing quality (AOQ) of these plans have been obtained under this assumption. It has been shown that, when the serial correlation coefficient of the Markov chain is positive, it is improper to use Dodge-type plans as the actual average outgoing quality limit (AOQL) in the plans for Markov-dependence exceeds the desired one under Dodge-type plans.

Nomenclature

X_n = 0 if the n^{th} unit produced is non-defective.
= 1 if the n^{th} unit produced is defective, $n \geqslant 0$.

D_i = 0 if the i^{th} unit produced is non-defective after the implementation of a plan.
= 1 otherwise, $i \geqslant 0$.

S_j = number of uninspected defective units passed in cycle j (cycle is the period where fractional sampling begins to the time it reverts to screening).

U_j = number of units undergoing screening from the end of cycle $(j-1)$ until the beginning of the success run of length r of non-defective units.

V_j = number of units passed under fractional sampling (whether inspected or not) in cycle j.

N_j = number of defective units undergoing screening from the end of cycle $(j-1)$ until the beginning of the success run of length r of non-defective units.

W_j = number of inspected defective units passed in cycle j.

$A(j)$ = serial number of the last unit in cycle j.

$p_{ij}^{(k)}$ = k-step transition probabilities of the Markov-chain $\{X_n, n \geqslant 0\}$ (see Feller[11] p. 432).

$(1-\delta)$= $[1-(\alpha-\beta)]$ ($= \lambda$ say) is the serial correlation coefficient of the Markov-chain.

p = $\alpha/(\alpha+\beta)$ is the long run proportion of defective units such that max $\{0, 1-\delta^{-1}\} < p < \min\{\delta^{-1}, 1\}$.

$A = p_{00}^{(k)}$, $B = 1-A$

$C = p_{11}^{(k)}$, $D = 1-C$

$E = p_{01}^{(k^2)}$

$G = q\,(1-p\,\delta)^{r-1}$

$H = 1-(1-\delta)^k$

$$A_1 = \sum_{h=1}^{k-1} p_{01}^{(h)} / p_{01}^{(k)}$$

$$A_2 = \sum_{h=1}^{k-1} p_{11}^{(h)}$$

$$A_3 = \sum_{h=1}^{k^2-1} p_{01}^{(h)} / p_{01}^{(k^2)}$$

1. Introduction

Continuous sampling plans find their applications in ammunition loading and component manufacture in the services of military. The plans are used when the production is continuous and the formation of inspection lots for lot-by-lot inspection may not be feasible as in conveyorised line production. Dodge[1], first devised a sampling inspection plan for continuous production, called continuous sampling plan—1 (CSP-1). The procedure of CSP-1 is as follows : At the start inspect 100 per cent of the units (screening) consecutively until r units in succession are found to be non-defective. When such a run of length r of non-defective units is observed, discontinue screening and inspect only a fraction $1/k$ of the units. If a defective unit is found revert immediately to screening.

Lieberman & Solomon[2] presented the theory of multi-level sampling plans (MLP), which is an extension of Dodge's work in single-level continuous sampling. The objectives of MLP were to permit a rapid reduction in inspection when quality was superior and to require screening only when the quality submitted was quite poor. The procedure of MLP is the same as CSP-1, except for the rule of action under fractional sampling which is as follows : When r inspected units under the $(i-1)^{th}$ level of sampling (0^{th} level is screening) is defect-free, switch over to i^{th} level of sampling ($i = 1, 2, \ldots, m$) if a defective unit is found within r inspected units under i^{th} level of sampling, revert to $(i-1)^{th}$ level ($i = 1, 2, \ldots, m$) of sampling. At level m, if r sampled units are defect-free, continue sampling at level m. The levels from two to six seem to be of practical interest.

All the continuous sampling plans assume that, a defective unit found during inspection are replaced by non-defective units. In certain conveyorized production line, such as explosives, it may not be possible to replace a defective unit by a non-defective unit, but it may be possible to remove it. Banzhaf & Brugger[3] and Brugger[4] discussed the application of continuous sampling plans (MIL-STD-1235 (ORD)) in U.S. military under a non-replacement assumption. Endres[5] discussed the computation of Unrestricted Average Outgoing Quality Limit (UAOQL) when defective unit is removed but not replaced. Abraham[6] gave a graphical method of parameter selection for three continuous sampling plans under a non-replacement assumption.

In almost all the continuous sampling plans, it is assumed that the production process is under statistical control (i.e., the probability of a defective unit does not change and the performance of a unit is not affected by any other unit). However, this assumption is not valid in practice. For example, when we are concerned with ordered observations, some kind of dependence is anticipated. Usually, the successive units in the production process tend to depend on each other. Here, we may recall the practical example of a scheme discussed by Broadbent[7] for Markov-dependent production process. He described a production process where a mould is continuously producing glass bottles in an automatic scheme from a single mould. He reported that, defective and non-defective bottles occur in runs and suggests therefore a Markov model.

Under the assumption that, the production process is not under statistical control, Lieberman[8], first presented an analysis of CSP-1. Under this assumption, the worst possible behaviour of the process would be to produce all non-defective units during screening and all defective units under fractional sampling. It is unrealistic that, an automated mass production would follow such a scheme. Dodge[9] and Sackrowitz[10] have contended that, this assumption is too unrestrictive and unrealistic.

Therefore, a mathematical model that compromises the statistical control situation and the total-lack of-control situation is to be sought. One such model that is mathematically feasible and includes the statistical control situation is the two-state time-homogeneous Markov-chain model.

The purpose of this paper is to extend CSP-1 and MLP with only two levels (MLP-2) to Markov-dependent production processes under a non-replacement assumption, using systematic sampling procedure (i.e., inspecting every k^{th} unit under fractional sampling). The implementation of these plans, technically assume that the serial correlation coefficient of the Markov-chain is known. The AOQL of these plans have been compared for different values of the serial correlation coefficient. We have demonstrated how the information obtained through the serial correlation coefficient of the Markov-chain is useful to a quality assurance practitioner to decide about the choice of a plan.

2. Assumptions

(i) We assume that $\{X_n, n \geqslant 0\}$ follows a $0 - 1$ valued Markov-chain with transition probability matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{bmatrix} 0 & 1 \\ 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} \quad 0 < \alpha,\ \beta < 1 \tag{1}$$

and initial distribution

$$P\,[X_0 = 1] = \pi_0,\ 0 \leqslant \pi_0 \leqslant 1 \tag{2}$$

(ii) We assume that $\pi_0 = 1$ (i.e., the zeroth unit is always defective) because screening is going to follow a defective unit in and cycle.

(iii) It is assumed that the zeroth unit is not counted in the AOQ.

Average Outgoing Quality and Average Outgoing Quality Limit

For any given plan (under a replacement assumption) it follows that the total number of uninspected defectives in m cycles, $S_1 + S_2 + \ldots + S_m = D_1 + D_2 + \ldots + D_{A(m)}$. From the properties of the Markov-chain $\{X_n, n \geqslant 0\}$ togethei with $\pi_0 = 1$, for $j \geqslant 1$ each of the sequences of random variables $\{S_j, j \geqslant 1\}$, $\{U_j, j \geqslant 1\}$, $\{N_j, j \geqslant 1\}$ and $\{W_j, j \geqslant 1\}$ is an independent and identically distributed (*i. i. d.*) sequence. Thus following Lieberman[8] the AOQ a plan (under a non-replacement assumption) is given by

$$\text{AOQ} = E\,(S_1)/\{\,[E\,(U_1+r) - E\,(N_1)] + [E\,(V_1) - E\,(W_1)]\} \tag{3}$$

It must be noted that the distribution of S_1, V_1 and W_1 would be different for different plans. However $E\,(U_1+r)$ and $E\,(N_1)$ are common to all the plans.

To derive $E\,(U_1 + r)$, we recall the definition of a failure sequence : a sequence is called a failure sequence if it contains r or less units and terminates with a defective unit. Noting that we are working $\pi_0 = 1$ we define P_1 as the probability of not finding r non-defective units before the first defective unit is found and h as the average number of units in a failure sequence. We find that,

$$h = \{(1-\alpha-\beta) + \beta\alpha^{-1}\,[1 - (1+r\alpha)\,(1-\alpha)^r]\}\ \ P_1^{-1}\,(1-\alpha)^{-1} \tag{4}$$

where

$$P_1 = (1-\beta)\,(1-\alpha)^{r-1} \tag{5}$$

It is easy to note that the probability of the number of failure sequences observed before getting a success sequence to be $P_1^j Q_1$ where $Q_1 = 1 - P_1$ and $j \geqslant 0$. Thus

$$E(U_1 + r) = h P_1 Q_1^{-1} + r \tag{6}$$

Substituting Eqns. (4) and (5) in Eqn. (6), we get finally after reparametrization

$$E(U_1 + r) = (1 - G)/pG\delta \tag{7}$$

When systematic sampling procedure is used, we have $P[V_1 = sk] = A^{s-1} B, s \geqslant 1$ so that

$$E(V_1) = k/p\,[1 - (1 - \delta)^k] \tag{8}$$

and $S_1 = V_1 \sum_{i=1}^{k^{-1}} T_i$, where T_i is the number of uninspected defective units in the i^{th} batch of k units in the first cycle. By Markov property T_i's are *i. i. d.* Hence by Wald's equation $E(S_1) = k^{-1} E(V_1) E(T_1)$ and $E(T_1) = \sum_{h=1}^{k-1} p_{01}^{(h)}$ so that

$$E(S_1) = \{k\delta - [1 - (1 - \delta)^k]\}/\delta\,[1 - (1 - \delta)^k] \tag{9}$$

It can be seen easily that

$$E(N_1) = P_1 Q_1^{-1} = (1 - G\delta)/G\delta \tag{10}$$

and

$$E(W_1) = 1 \tag{11}$$

Substituting Eqns. (7)–(11) in Eqn. (3), we obtain the AOQ expression for CSP-1 under a non-replacement assumption as

$$\text{AOQ} = pG(k\delta - H)/[Hq + G(k\delta - H)]. \tag{12}$$

To obtain the AOQ expression for MLP-2 for Markov-dependent production processes, it is enough if we compute $E(V_1)$, $E(S_1)$ and $E(W_1)$. The expressions $E(U_1 + r)$, $E(N_1)$ remain the same as in Eqns. (7) and (10). Recalling the procedure of MLP-2, we now have the probability of possible cycles and the average number of units passed under fractional sampling as follows :

Probability of possible cycles	Average number of units passed under fractional sampling
$A^{i-1} B,\ i = 1, 2, \ldots, r$	$\sum_{i=1}^{r} BA^{i-1}\, ik$
$A^r (DA^{r-1})^{t-1} C,\ t \geqslant 1$	$\sum_{t=1}^{\infty} A^r (DA^{r-1})^{t-1} C\,[(v + rk)\,t + k]$
$A^r (DA^{r-1})^{t-1} DA^{i-2} B,\ t \geqslant 1$ and $i = 2, 3, \ldots, r$	$\sum_{t=1}^{\infty} \sum_{i=1}^{r} A^r (DA^{r-1})^{t-1} DA^{i-2} B\,[(v + rk)\,t + ik]$

where $v = k^2/E$ is the average number of units passed from the end of first level of fractional sampling until a defective unit is found under the second level of fractional sampling. Now,

$$E(V_1) = \sum_{i=1}^{r} A^{i-1} B_{ik} + \sum_{t=1}^{\infty} A^r (DA^{r-1})^{t-1} C[(v + rk)t + k]$$

$$+ \sum_{t=1}^{\infty} \sum_{i=2}^{r} (DA^{r-1})^{t-1} DA^{i-2} B[(v + rk)t + ik]$$

$$= \{k[1 - A^r + BA^{r-1}(AC - BD)] E + A^r Bk^2\}/EB(1 - DA^{r-1}) \quad (13)$$

$$E(S_1) = \sum_{i=1}^{r} [A^{i-1} B] i BA_1 + \sum_{t=1}^{\infty} A^r [DA^{r-1}]^{t-1} C$$

$$[\{BA_1 + (t-1)[A_3 + A_2 + (r-1) BA_1]\} + A_3 + A_2]$$

$$+ \sum_{t=1}^{\infty} \sum_{i=2}^{r} A^r [DA^{r-1}]^{t-1} DA^{i-2} B[\{BA_1 + (t-1)[A_3 + A_2$$

$$+ (r-1) BA_1]\} + A_3 + A_2 + BA_1 i]$$

$$= \{A^r [A_3 + A_2(C + DA^{r-1}) - A_1(1 + D^{r-1} B)]$$

$$+ A_1(1 - DB^2 A^{r-1})\}/(1 - DA^{r-1}) \quad (14)$$

$$E(W_1) = [1 - J A^r (2 - J)]/(1 - J), \text{ where } J = DA^{r-1} \quad (15)$$

Substituting Eqns. (7), (10), (13), (14) and (15) in Eqn. (3), we obtain the AOQ for MLP-2 under a non-replacement assumption as

$$\text{AOQ} = \frac{BEpG\delta \{A_1(1 - JB^2) + A^r [A_3 + A_2(J + C) - A_1(1 + JB)]\}}{\{[q - G(1 - p\delta)](1 - J) BE + pG\delta R\}} \quad (16)$$

where

$$R = \{KE[1 - A^r + BA^{r-1}(AC - BD)] + A^r K^2 B - BE[1 - J + (2 - J) A^r]\}$$

When the serial correlation coefficient is known, it is possible to obtain a relationship between r, k and AOQL (p_L) in CSP-1 for Markov-dependent production processes (like the one given by Dodge[1]).

The relationship between r, k and p_L in CSP-1 for Markov-dependent production process under a non-replacement assumption is obtained as $p_L = p\{1 + H(1 - p) G^{-1} I^{-1}\}^{-1}$, where p is the solution of the equation

$$p = [p_L \delta (r + 1) + 1] (r\delta)^{-1}$$

In MLP-2 for Markov-dependent production process, such relationship between r, k and AOQL cannot be obtained by analytical methods. Therefore, we use the computer to obtain the AOQL values. For fixed r, k and known δ we calculate the

AOQ values for $p = 0.001250, 0.00250, \ldots$etc., bearing the mind that $\max\{1-\delta^{-1}, 0\} < p < \min\{\delta^{-1}, 1\}$. The computer programme reports only that value of the AOQ for which the next calculated value is smaller. This is justified because the AOQ function in Eqn. (16) is continuous and concave. In computing the AOQL values we have used an iterative procedure. After getting extensive tables giving approximate AOQL values for the triplet (r, k, δ) we collect certain AOQL values which are commonly used.

3. Discussion

We first of all observe from the Table 1 that, when the production process is not under statistical control, the actual AOQL obtained under the plan following Markov-dependence with serial correlation coefficient positive, always exceeds the AOQL that is desired under statistical control situation. However, when the serial correlation coefficient is negative the desired AOQL is always guaranted by the plans that are under statistical control. We also note from the Table 1 that, MLP-P-2 always carry higher AOQL values as compared to CSP-1. However, when p remains constant, the limiting value of AOQ as $\delta \to 0$ is zero in CSP-1 and MLP-2 plans. Similarly the limiting value of AOQ as $\delta \to 2$ is zero in both the plans. Therefore, we find that CSP-1 gives more protection in terms of AOQL than MLP-2.

Table 1. Comparison of AOQL values in percentage for $k = 7$ and $r = 43$

Plans / λ	0.91	0.74	0.46	0.31	0.09	0.05	0.00	−0.91	−0.82
CSP−1	2.64	2.65	2.49	2.37	2.17	2.13	2.09	0.89	0.0248
MLP−2	3.38	3.23	3.26	3.19	3.04	3.00	2.97	1.04	0.0250

Thus when the serial correlation coefficient is positive, the desired AOQL cannot be attained when Dodge-type plans are used. At the same time, when the serial correlation coefficient is negative, the Dodge-type plans can still be used as the AOQL under Dodge-type plans always guarantee the decided AOQL (because p cannot exceed $\min\{\delta^{-1}, 1\}$). The AOQ expressions for CSP-1 and MLP-2 for Markov-dependent process under replacement assumption has been obtained in Rajarshi & Sampath Kumar[12].

4. Conclusion

When the successive units in a continuous production process is neither under a state of statistical control nor follow a scheme of total-lack of-statistical control, the two-state time-homogeneous Markov-chain model that compromises both the situations seem to be quite appropriate. It is concluded that, when the serial correlation coefficient of the Markov-chain is positive, CSP-1 and MLP-2 plans which assume statistical control should not be used because the actual AOQL given by Markov-dependent process always exceeds the desired AOQL. Under Markov-dependence, CPS-1 may be preferred to MLP-2 as the former gives more protection in terms of AOQL than the latter.

Finally, it is recommended that the quality assurance practitioner should carry out a test for independence (see Lehmann[13] p. 155) and decided ultimately to choose a plan.

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