Def Sci J, Vol 33, No 4, October 1983, pp 289–297

Cylindrical Shock in a Self-Gravitating Gas

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Received 14 July 1983

Abstract. The propagation of diverging cylindrical shock in a self-gravitating gas having an initial density distribution $\rho_0 = \beta r^{-w}$, where β and w are constants, has been studied for the two cases : (i) when the shock is strong and (ii) when it is weak. Analytical relations for shock velocity and shock strength have been obtained. It is shown that for strong shock, the shock velocity must decrease continuously with shock propagation; for weak shock, it initially decreases and attains a minimum value for certain propagation distance $r_{U_{min}}$ and beyond this distance starts increasing.

Finally, the expressions for the pressure, the density and the particle velocity immediately behind the shock have been obtained for both the cases.

1. Introduction

The non-linearity of shock phenomenon is retained upto several hundred meters and the similarity method is inadequate in a region where Mach number M is taken as

$$M = 1 + \epsilon, \tag{1}$$

where ϵ is a parameter which is negligible in comparison to unity. Recently, Chisnell-Chester-Whitham[1–3] method has been used to investigate the propagation of weak shocks[4].

In the present paper CCW method is applied to study the propagation of diverging cylindrical shock in a self-gravitating gas having an initial density distribution $\rho_0 = \beta r^{-w}$, simultaneously for the two cases : (i) when the shock is strong and (ii) when it is weak. Analytical relations for shock velocity and shock strength have been obtained. The expressions for the pressure, the density and the particle velocity immediately behind the shock have also been derived.

The results accomplished, have been compared through Figures and Tables.

2. Basic Equations, Boundary Conditions and Analytical Expressions for Shock Velocity

The equations governing the cylindrically symmetrical flow of the gas under the influence of its own gravitation are

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{Gm}{r^2} = 0 \\ &\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) = 0 \\ &\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r} - a^2\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r}\right) = 0 \\ &\frac{\partial m}{\partial r} = 2\pi r\rho \end{aligned}\right\} \quad (2)$$

where $m(r, t)$, $u(r, t)$, $p(r, t)$ and $\rho(r, t)$ denote respectively the mass inside a cylinder of radius r and unit length, the velocity, the pressure and the density at a distance r at time t and $a^2 = \gamma p/\rho$.

Let p_0 and ρ_0 denote the undisturbed values of pressure and density in front of the shock wave, and u_1, p_1 and ρ_1 be the values of the respective quantities at any point immediately, after the passage of the shock, then the well known Rankine-Hugoniot conditions permit us to express u_1, p_1 and ρ_1 in terms of the undisturbed values of these quantities by means of the following equations:

$$\begin{aligned} p_1 &= \rho_0 a_0^2 \left\{\frac{2}{(\gamma+1)} M^2 - \frac{(\gamma-1)}{\gamma(\gamma+1)}\right\} \\ \rho_1 &= \rho_0 \frac{(\gamma+1) M^2}{(\gamma-1) M^2 + 2} \\ u_1 &= \frac{2a_0}{(\gamma+1)}\left(M - \frac{1}{M}\right) \\ U &= a_0 M, \end{aligned} \quad (3)$$

where U is the shock velocity and a_0 is sound velocity in undisturbed medium.

Case 1. For strong shock, $U \gg a_0$, then boundary conditions (3) reduce to

$$p_1 = \frac{2\rho_0}{\gamma+1} U^2, \quad \rho_1 = \rho_0 \frac{(\gamma+1)}{(\gamma-1)} \text{ and } u_1 = \frac{2U}{(\gamma+1)} \quad (4)$$

Case 2. For weak shock, using relation (1) boundary conditions (3) become

$$\begin{aligned} p_1 &= \frac{\gamma p_0}{(\gamma+1)}\left\{\frac{(\gamma+1)}{\gamma} + 4\epsilon\right\} \\ \rho_1 &= \rho_0\left\{1 + \frac{4\epsilon}{(\gamma+1)}\right\} \end{aligned} \quad (5)$$

$$u_1 = \frac{4a_0\epsilon}{(\gamma + 1)}$$

For diverging shock the characteristic form of system of equations (1) that is, the form in which equation contains derivatives in only one direction in (r, t) plane, is

$$dp + \rho\, a\, du + \frac{(\rho a^2 u)}{(u + a)} \frac{dr}{r} + \frac{\rho a\, Gm}{(u + a)} \frac{dr}{r^2} = 0 \qquad (6)$$

The final step is to substitute the shock conditions (4) or (5) into this relation. A first order differential equation in U^2 or $\epsilon\,(r)$ is obtained which determines the shock. For strong shock substituting (4) into Eqn. (6), we get

$$\frac{dU^2}{dr} + \frac{A}{r} U^2 = -\frac{B\, Gm}{r^2}, \qquad (7)$$

where

$$A = 4\gamma\,[\,\{2 + \sqrt{2\gamma\,(\gamma - 1)}\}\,\{2\gamma/(\gamma - 1) + \sqrt{2\gamma/(\gamma-1)}\}\,]^{-1}$$

$$B = [(\gamma + 1)^2 \sqrt{2\gamma\,(\gamma - 1)}]\,[(\gamma - 1)\,\{2 + \sqrt{2\gamma\,(\gamma - 1)}\}\,\{2\gamma/(\gamma - 1) + \sqrt{2\gamma/(\gamma - 1)}\}]^{-1}$$

On integration Eqn. (7) yields,

$$U^2 = r^{-A}\,\{K - BG \int m\, r^{A-2}\, dr\}, \qquad (8)$$

where K is a constant of integration.
Now, assuming the initial density distribution law as $\rho_0 = \beta r^{-w}$, the mass inside a cylinder of radius r and unit length, is written as

$$m = \frac{2\pi\,\beta}{2 - w}\, r^{2-w} \qquad (9)$$

Substituting the value of m in Eqn. (8) and integrating, we get

$$U^2 = r^{-A}\,\{K - C r^{A+1-w}\} \qquad (10)$$

where $C = \dfrac{2\pi\beta\, GB}{(2 - w)\,(A + 1 - w)}$

The condition of hydrostatic equilibrium prevailing in front of shock is written as,

$$\frac{1}{\rho_0} \frac{dp_0}{dr} = -\frac{Gm}{r^2}. \qquad (11)$$

Now for weak shock, using conditions (5) and substituting into (6) and remembering (11) also, we get

$$d\epsilon + \tfrac{1}{2}\left(\frac{dp_0}{p_0} + \frac{da_0}{a_0} + \frac{dr}{r}\right)\epsilon = 0, \qquad (12)$$

which immediately integrates to

$$\epsilon = K_1 \, p_0^{-1/2} \; a_0^{-1/2} \; r^{-1/2}, \tag{13}$$

where K_1 is a constant of integration.

From Eqn. (11), the pressure p_0 and sound velocity a_0 can be written as

$$p_0 = \frac{2\pi\beta^2 G\, r^{-2w+1}}{(2w-1)(2-w)} = K_2\, r^{-2w+1}$$

and

$$a_0 = \left[\frac{2\pi\beta\, G\gamma}{(2w-1)(2-w)}\right]^{1/2} r^{1-w/2} = K_3\, r^{\frac{1-w}{2}}$$

where

$$K_2 = 2\pi\, \beta^2 G/(2w-1)(2-w)$$

$$K_3 = \sqrt{2\pi\beta\, G\gamma/\,(2w-1)(2-w)}$$

Positivity and finiteness of the equilibrium pressure as defined by the above equation, requires that constant w should obey the inequality,

$$\tfrac{1}{2} < w < 2 \tag{15}$$

Substituting the values from Eqn. (13) in Eqn. (12), we get

$$\epsilon = K_4\, r^{\frac{5}{4}(w-1)} \quad \text{where } K_4 = K_1\, K_2^{-1/2}\, K_3^{-1/2}$$

This value of ϵ together with Eqn. (5) gives for weak shock

$$U = K_3\, r^{\frac{1-w}{2}} + K_3\, K_4\, r^{\frac{3}{4}(w-1)} \tag{16}$$

Now the expressions for shock strength can be easily written as

$$\left(\frac{U}{a_0}\right)^2 = \frac{(2w-1)(2-w)}{2\pi\,\gamma\,\beta\, G}\, K\, r^{(-A+w-1)} - C \tag{17}$$

and

$$\frac{U}{a_0} = 1 + K_4\, r^{\frac{5}{4}(w-1)} \tag{18}$$

respectively for strong and weak shock.

The pressure, the density and the particle velocity immediately behind the shock for the two cases are given as,

$$p = \frac{2\beta r^{-w-A}}{\gamma+1}(K - C\, r^{A-w+1}),$$

$$\rho = \frac{\gamma+1}{\gamma-1}\,\beta\, r^{-w}, \qquad (19)$$

$$u = \frac{2}{(\gamma+1)}\, r^{-A/2}\,\{K - C\, r^{A-w+1}\}^{1/2}$$

and

$$p = K_2\, r^{-2w+1} + \frac{4\gamma}{\gamma+1} K_2\, K_4\, r^{-\frac{3w+1}{4}},$$

$$\rho = \beta\, r^{-w} + \frac{4\beta}{\gamma+1} K_4\, r^{\frac{w-5}{4}} \qquad (20)$$

$$u = \frac{4K_3\, K_4}{\gamma+1}\, r^{\frac{3}{4}(w-1)}$$

3. Discussions

The Eqn. (10) for shock velocity representing the propagation of strong diverging cylindrical shock through self-gravitating gas indicates that as the shock advances, the shock velocity must decrease. General consequences of Eqn. (17) are that (i) the shock will be relatively strengthened or weakened as w is greater or less than $1 + A$, and (ii) the case $w = 1 + A$ corresponds to the shock moving with constant strength, i. e, to a shock whose radius increases uniformly for all times. Any such case is, however, astrophysically important because of the fact that the radial velocity observations of most *novae* in the course of their outbursts indicate velocities of expansion which-different as they may be from star to star remains more or less constant within periods of the order of several days. An increase in w from 1/2 to $1 + A$ reduces the shock velocity whereas for further increase in w from $1 + A$ to 2, the shock velocity increases (Ref. Table I).

Variation of shock velocity and shock strength with propagation distance for $w = 1.05$, 1.18, 1.2 and 1.95 have been shown in Fig. 1, taking $\frac{U}{a_0} = 20$ at $r = 1.0$. Strengthening for $w = 1.05$ and 1.18 and weakening for $w = 1.2$ and 1.95 is evident.

Equation (16) contains two terms involving the propagation distance r one with positive power and other with negative power of r for all values of w lying in the range $\frac{1}{2} < w < 2$ except $w = 1$. Whereas for low values of r the one with negative power happens to be dominant term, for large values of r, it is the other term which primarily determines the shock velocity. Consequently, the shock velocity

Table 1. Variation of shock velocity, shock strength and density with distance for strong cylindrical shock

	$W = 1.15$			$W = 1.8$		
r	U/a_0	$U/\sqrt{\beta G}$	ρ_0/β	U/a_0	$U/\sqrt{\beta G}$	ρ_0/β
1.00	20.0000	56.440413	1.0000	20.0000	82.275332	1.0000
1.02	19.991876	56.333758	0.9774843	20.120718	82.118894	0.964983
1.04	19.983912	56.22937	0.9558982	20.239807	81.965806	0·931837
1.06	19.976104	56.127151	0.9351865	20.357317	81.815932	0.9004289
1.08	19.968444	56.027035	0.9152983	20.4733	81.669147	0.8706372
1.10	19.960928	55.928924	0.8961865	20.587801	81.52533	0.8423511
1.12	19.953549	55.832748	0.8778074	20.700867	81.384369	0.8154691
1.14	19.946304	55.738434	0.8601207	20.812541	81.246159	0.7898984
1.16	19.939187	55.645915	0.8430887	20.922863	81.110597	0.7655536
1.18	19.932194	55.555127	0.8266766	21.031874	80.97759	0.7423562
1.20	19.92532	55.466009	0.8108519	21.13961	80.847045	0.7202342

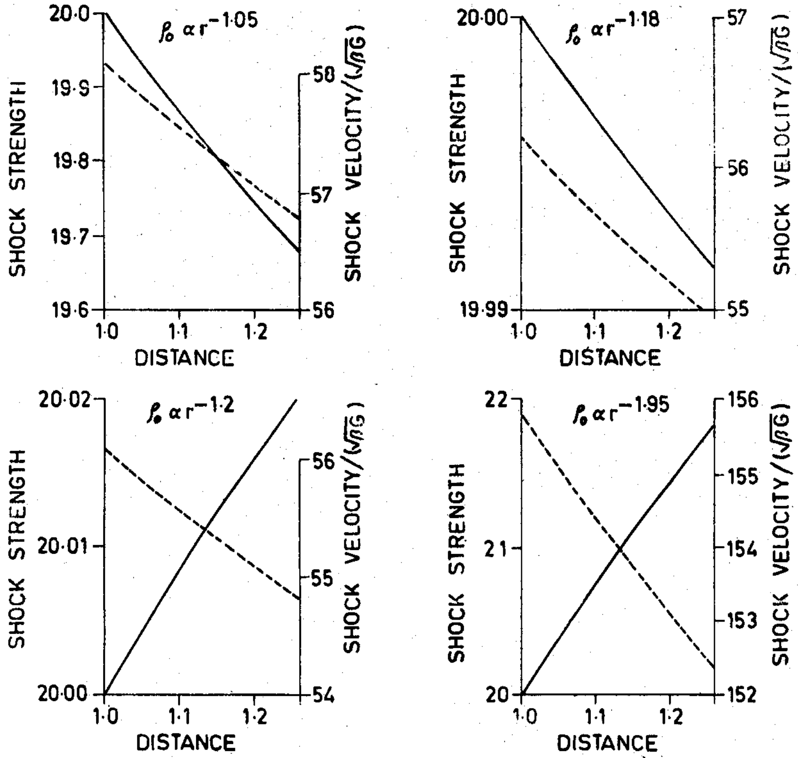

Figure 1. Variation of shock strength (—) and shock velocity (...) with distance for strong shocks.

Table 2. Variation of $r_{U_{\min}}$ and shock velocity with w

W	K_4	$r_{U_{\min}}$	U/K_3	U/a_0
0.75	1.0542412	4.3341105	2.0019869	1.6666666
1.05	0.1874735	6.5388×10^8	1.0033702	1.6666663
1.15	0.1054241	18699.664	0.7970053	1.6666666
1.18	0.0887033	7819.7954	0.7438055	1.6666666
1.6	0.0079056	369.79855	0.2827868	1.6666666
1.8	0.0025	266.66666	0.1784272	1.6666666
1.95	0.0010542	228.4151	0.1263148	1.6666666

initially decreases as the shock progresses and attains a minimum value for a certain propagation distance $r_{U_{\min}}$ given by

$$r_{U_{\min}} = \left[\frac{2}{3K_4} \right]^{4/5(w-1)}$$

Table 3. Variation of shock velocity and shock strength with distance for weak shocks

$W = 1.05$			$W = 1.18$			$W = 1.8$		
r	U/K_3	U/a_0	r	U/K_3	U/a_0	r	U/K_3	U/a_0
10	1.1484415	1.216491	2	1.0369272	1.1036743	50	0.2352688	1.1250
50	1.1239254	1.2394007	10	0.9338735	1.1489154	75	0.2111587	1.1875
100	1.1140636	1.2499999	25	0.8854703	1.1830106	100	0.1981116	1.2500
1000	1.084302	1.2886954	100	0.8258667	1.2499998	200	0.1801686	1.5000
10000	1.0591415	1.3333802	500	0.7768589	1.3590939	250	0.178516	1.6250
500000000	1.003404	1.6555797	1000	0.7624244	1.4197007	265	0.1784281	1.6625
750000000	1.003379	1.6724054	7500	0.7438164	1.6604326	275	0.1784475	1.6875
1000000000	1.0034552	1.6846047	10000	0.7440828	1.7045953	300	0.1787267	1.7500

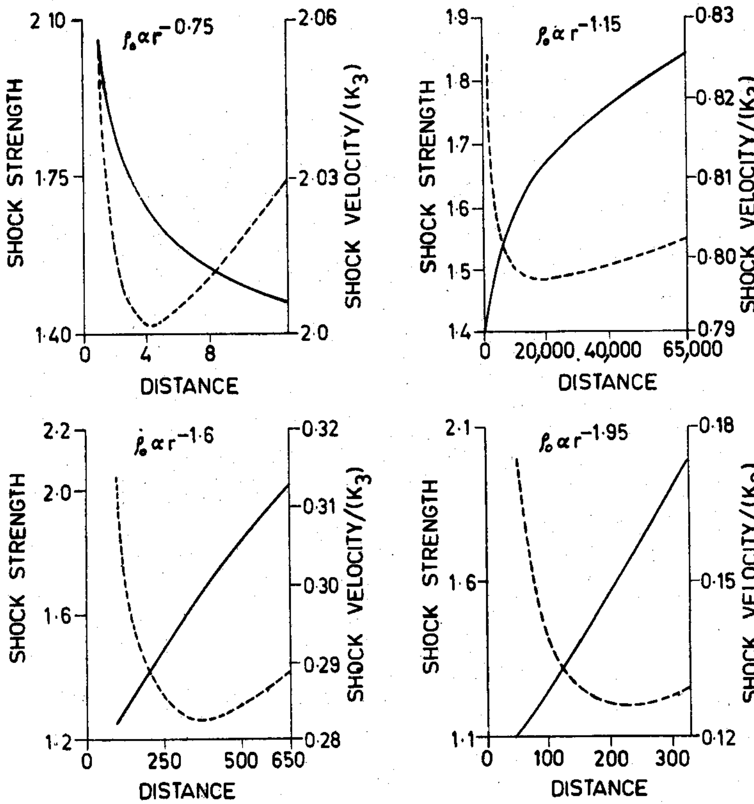

Figure 2. Variation of shock strength (—) and shock velocity (...) with distance for weak shocks.

which depends on initial strength of the shock. Beyond this distance shock velocity starts increasing. Similar variations in shock velocity for spherical shock through self-gravitating gas has been reported by Kumar and Saxena[5]. Taking $\frac{U}{a_0} = 1.25$ at $r = 100$, $r_{U_{\min}}$ together with minimum value of the shock velocity for different values of w are given in Table 2. An increase in w reduces the shock velocity (Ref. Table 3). Fig. 2 shows the variation of shock velocity with propagation distance for $w = 0.75$, 1.15, 1.6 and 1.95. Case $w = 0.75$ weakens the shock whereas for the other values the shock is strengthened.

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