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Unsteady Viscous Flow Down an Inclined Open Channel with Naturally Permeable Bed

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Abstract. The exact velocity distribution for the flow of viscous incompressible fluid in an inclined channel with naturally permeable bed has been obtained by using the Laplace transform and the Finite Fourier Sine transform technique. It has been assumed that (i) the fluid flows in the steady state for $t \leqslant 0$, (ii) unsteady state occurs at $t > 0$, and (iii) unsteady state motion is influenced by time dependent pressure gradient.

1. Introduction

The open channel flow has an important application in the field of hydraulic engineering. Several empirical results have been reported by many investigators, e.g. Vanoni[1] and Johnson[2]. Verma & Vyas[3] derived analytically the results for velocity distribution and flux under steady state of flow down an open channel with permeable bottom and impermeable vertical walls. In this paper, we have discussed the unsteady state of the same flow under arbitrary pressure gradient.

2. Formulation of the Problem

Let us consider the flow of fluid of density ρ and viscosity μ in an open inclined channel of width $2a$ and depth h having its side walls normal to the bottom. The plane of the bottom (naturally permeable bed) is inclined at an angle ($\beta \leqslant \pi/2$) with the horizontal. x-axis is taken in the central line along the direction of flow at the free surface which is stress free, y-axis along the depth and z-axis along the width of the channel.

The governing equations of motion for the viscous incompressible fluid flowing down an open inclined channel at $t > 0$ are

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \rho g \sin \beta + \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \tag{1}$$

$$0 = \frac{\partial p}{\partial y} + \rho g \cos \beta, \tag{2}$$

$$0 = \frac{\partial p}{\partial z}, \tag{3}$$

$$\frac{\partial u}{\partial x} = 0. \tag{4}$$

Introducing the dimensionless quantities

$$x' = \frac{x}{h},\ y' = \frac{y}{h},\ z' = \frac{z}{h},\ t' = \frac{t\nu}{h^2},\ u' = \frac{u}{U},\ K' = \frac{K}{h^2},$$

$p' = \dfrac{p}{\rho U^2}$, where U being the characteristic velocity.

The Eqn. (1) gives (after dropping dashes)

$$\frac{\partial u}{\partial t} = -R \frac{\partial p}{\partial x} + \frac{R \sin \beta}{F} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \tag{5}$$

where R (Reynold's number) $= \dfrac{Uh}{\nu}$ and F $\left(\text{(Froude number} = \dfrac{U^2}{gh}\right)$.

Initial and boundary conditions accordingly are

For $t > 0$,

$$\left.\begin{array}{ll} \text{(i)} & u = 0 \text{ at } z = \pm \dfrac{a}{h} = \pm l, \\ \text{(ii)} & \dfrac{\partial u}{\partial y} = 0 \text{ at } y = 0. \\ \text{(iii)} & u = Q_1 - \dfrac{\sqrt{K}}{\alpha} \dfrac{\partial u}{\partial y} \text{ at } y = 1 \text{ (Beavers \& Joseph}^4\text{)}, \end{array}\right\} \tag{6}$$

where $Q_1 = K R \left(-\dfrac{\partial p}{\partial x} + \dfrac{\sin \beta}{F} \right)$, K is permeability of bed and α is a dimensionless quantity depending upon porous material.

For $t \leqslant 0$,

$u = u_0$, u_0 being steady state velocity.

3. Solution

Let

$$-\frac{\partial p}{\partial x} = p_0 \text{ for } t \leqslant 0,$$
$$= g(t) \text{ for } t > 0.$$

and taking $z = \frac{2l\xi}{\pi} - l$, Eqn. (3) becomes

$$\frac{\partial u}{\partial t} = R\,g(t) + \frac{R \sin \beta}{F} + \frac{\partial^2 u}{\partial y^2} + \frac{\pi^2}{4l^2} \frac{\partial^2 u}{\partial \xi^2} \text{ at } t > 0 \tag{7}$$

and u_0 is given by (Verma and Vyas[3]),

$$u_0 = \frac{2}{\pi} \sum_{n=1}^{\infty} N \left\{ U_1 \frac{\cosh Qy}{\cosh Q} + \frac{p}{Q^2} \left(1 - \frac{\cosh Qy}{\cosh Q} \right) \right\} \sin n\,\xi \tag{8}$$

where

$$N = \frac{1 - \cos n\pi}{n}, \quad P = \rho_0 + \frac{\sin \beta}{F}; \quad Q^2 = \frac{n^2 \pi^2}{4\,l^2}$$

and

$$U_1 = \frac{R\,P \left\{ K + \frac{\sqrt{K}}{\alpha} \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{N}{Q^2} \tanh Q \sin n\,\xi \right\}}{1 + \frac{\sqrt{K}}{\alpha} \frac{2}{\pi} \sum_{n=1}^{\infty} N\,Q \tanh Q \sin n\,\xi}.$$

To solve the Eqn. (7), we shall make use of the following transforms

(1) Laplace transform with respect to t (Sneddon[5], p. 136) defined as

$$\bar{u}(y, \xi, s) = \int_0^{\infty} u(y, \xi, t)\, e^{-st}\, dt, \; s > 0 \tag{9}$$

with condition $u = u_0$ at $t = 0$.

(2) Finite Fourier Sine transform (Sneddon[5], p. 426) defined as

$$\bar{u}^*(y, n, s) = \int_0^{\pi} \bar{u}(y, \xi, s) \sin n\,\xi \tag{10}$$

with condition $\bar{u} = 0$ at $\xi = 0$ and $\xi = \pi$.

Now taking Laplace transform (9) and then Finite Fourier—Sine transform (10) on Eqn. (7) and then solving the equations so obtained under the conditions,

$$\frac{\partial \bar{u}^*}{\partial y} = 0 \text{ at } y = 0$$

$$\bar{u}^* = K\,R \left\{ \bar{g}(s) + \frac{\sin \beta}{F\,s} \right\} N - \frac{\sqrt{K}}{\alpha} \frac{\partial \bar{u}^*}{\partial y} \text{ at } y = 1$$

We obtain

$$\bar{u}^* = N \left[R\,K \left\{ \bar{g}(s) + \frac{\sin \beta}{F\,s} \right\} \frac{\cosh Hy}{\cosh H + \frac{H\sqrt{K}}{\alpha} \sinh H} \right.$$

$$+\frac{R}{H^2}\left\{\bar{g}(s)+\frac{\sin\beta}{Fs}\right\}\left\{1-\frac{\cosh Hy}{\cosh H+\frac{H\sqrt{K}}{\alpha}\sinh H}\right\}$$

$$+\frac{1}{s}(P_1-U_1)\left\{(1+T)\frac{\cosh Hy}{\cosh H+\frac{H\sqrt{K}}{\alpha}\sinh H}\frac{\cosh Qy}{\cosh Q}\right\}$$

$$\left.+\frac{P_1}{H^2}\left(1-\frac{\cosh Hy}{\cosh H+\frac{H\sqrt{K}}{\alpha}\sinh H}\right)\right] \tag{11}$$

where $H^2=s+Q^2$, $T=\frac{\sqrt{K}}{\alpha}Q\tanh Q$, $P_1=\frac{P}{Q^2}$.

Now inverting the Finite Fourier transform in Eqn. (11) by using (Sneddon[5], p. 426) and then inverting the Laplace transform by using inversion theorem (Sneddon[5], p. 174) and also using the Convolution theorem (Sneddon[5], p. 171), the velocity distribution is finally obtained as

$$u=\frac{2}{\pi}\sum_{n=1}^{\infty}N\sin n\xi\left[U_1\sum_{r=0}^{\infty}\frac{\cos a_r v.\ 2\alpha^2.\ a_r\ e^{-(Q^2+a_r^2)t}}{(Q^2+a_r^2)\ S_n}\right.$$

$$+P_1\sum_{r=0}^{\infty}\frac{\cos a_r y.\ 2\alpha^2\ Q^2\ e^{-(Q^2+a_r^2)t}}{(Q^2+a_r^2)\ .\ S_n\ a_r}+T(U_1-P_1)$$

$$\left.\sum_{r=0}^{\infty}\frac{\cos_2 a_r y.\ 2\alpha^2.\ a_r\ e^{-(Q^2+a_r^2)t}}{(Q^2+a_r^2)\ S_n}+R\int_0^t h(\lambda).\ g(t-\lambda)\,d\lambda\right] \tag{12}$$

where $S_n=\sin a_r\ (\alpha^2+\alpha\sqrt{K}+K a_r^2)$.

a_r = the root of $a_r\tan a_r=\frac{\alpha}{\sqrt{K}}$.

and

$$h(\lambda)=\sum_{r=0}^{\infty}\frac{\cos a_r y\ 2\alpha^2}{(\alpha^2+\sqrt{K}\alpha+K a_r^2\sin a_r}\left\{K a_r+\frac{1}{a_r}\right\}e^{-(Q^2+a_r^2)\lambda},$$

4. Discussion

Equation (12) defines the velocity distribution when the pressure gradient is an arbitrary function of time and from it various cases can be deduced for different forms of $g(t)$. Graphs have been drawn to represent the velocity profiles when $g(t)=P_0e^{-bt}$, it is observed that

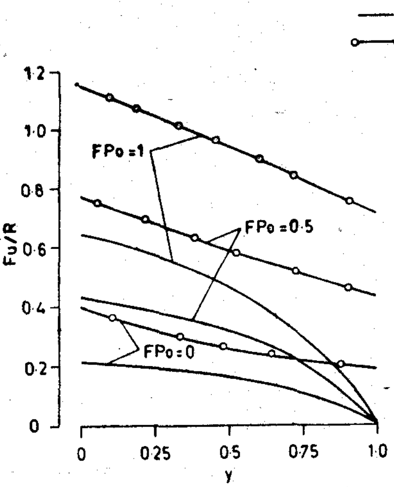

Figure 1. Velocity plotted against y when $g(t) = P_0 e^{-bt}$.

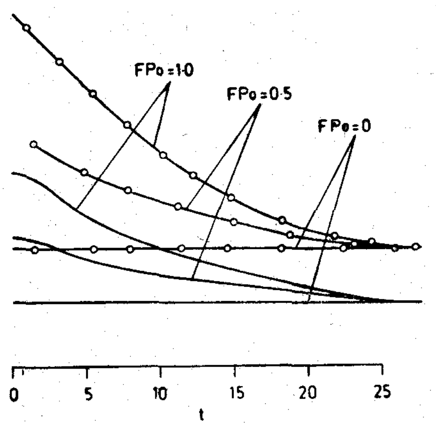

Figure 2. Velocity plotted against t wnen $g(t) = P_0 e^{-bt}$.

(1) The flow increases when the bottom of the channel is permeable in comparison to the impermeable bottom, whatever be the pressure gradient (Figs. 1 and 2). This is on account of the slip condition on the permeable bed.

(2) With the increase in time the velocity decreases in such a manner that the steady state is again reached for large values of $t > 25$, (Fig. 2).

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