Def Sci J, Vol 33, No. 2, April 1983, pp 127–135

Unsteady Compressible Stokes Layers on a Disk

K. N. Venkatasiva Murthy
Department of Mathematics, S. K. University, Anantapur-515 003

Received 17 July 1981; revised 12 October 1982

Abstract. The unsteady motion of a viscous compressible rotating gas bounded by a single infinite disk is studied when small amplitude torsional oscillations are superimposed on the disk. The generation and propagation of waves due to the interactions of compressibility, viscosity and rotation during the transient evolution is discussed. In comparison with the non-oscillatory case, the forcing frequency of the disk is capable of inducing several additional interesting features.

1. Introduction

The dynamics of homogeneous rotating fluids is basically concerned with the motion of a fluid bounded by one or two disks, as the fluid adjusts from one state of rigid rotation to another state of rigid rotation or oscillation about a state of rigid rotation. Such models provide valuable insight into physical problems which are of interest in astrophysics and geophysics. The relevance of rotating fluid theory to physical situations like solar spin down is well known and described by Howard etal[1]. The theory of compressible rotating gases is important in relation to gas centrifuges used for the enrichment of uranium and has applications in nuclear physics. In view of these applications, the theory of rotating fluids is useful in defence science.

Viscous incompressible flows due to oscillating disk or disks are well known in literature[2,3&4]. These works deal with the study of interactions between viscous diffusion, Coriolis force and inertial oscillations leading to the formation of modified Stokes layers on a single disk or two disks seperated by a finite distance. Unsteady motion of a compressible gas has recently been investigated by Iwao Harada[5] wherein the motion is created by subjecting the angular velocity of the disk to a small impulsive change. The effect of compressibility on inertial oscillations and the unsteady Ekman layer are investigated.

It is well known that the rigid rotation of the fluid provides a restoring mechanism in the fluid and makes wave motion possible. It is also well known that viscosity provides a source of diffusivity and results in the propagation of diffused waves in the fluid. Thus it will be of importance and interest to investigate the rich properties of wave motion possible in a compressible rotating gas. Infact, the problem described by Iwao Harada[5] itself needs a reconsideration in exploring the wave mechanism.

Motivated by these aspects, the physical problem concerning the unsteady motion of a compressible rotating gas created by superimposing torsional oscillations on the disk is studied. The aim of the paper is to explore the wave phenomena during the transient time in addition to the usual investigations on the boundary layer and the flow outside. It is established that the unsteady flow is marked by the propagation of shear type decaying wave trains generated by the triangular interaction between diffusion, compressibility and inertial oscillations. The torsional oscillations imposed on the disk bring into effect some additional wave modes and additional characters which will be absent either in the non-oscillatory case or in the case of an incompressible fluid. Also, the compressibility and the forcing oscillations manifest together in introducing multiple boundary layers on the disk.

2. Formulation

A compressible gas is assumed to fill the space above an infinite disk situated at $z' = 0$ with reference to cylindrical polar coordinates (r', ϕ', z'). Prior to time $t' = 0$, the whole system is in a state of rigid rotation at angular speed Ω about the z' axis. At time $t' = 0$, the boundary angular speed is impulsively changed to $\Omega + \beta\, \Delta\Omega$ exp (in 't''). The disk acts as a sink or source to the gas according as $\beta > 0$ or $\beta < 0$. The characteristic values of length, velocity, time, temperature and density are taken as L, ΩL, Ω^{-1}, T_0 and ρ_c. Assuming small values of the Rossby number $\epsilon = \Delta\Omega/\Omega$, the variables are expanded in series of powers of ϵ. If the non-dimen sional velocity components of the fluid measured relative to a rotating frame with angular velocity $\bar{\Omega}$ are (u^*, v^*, w^*), we write

$$\rho^* = \rho_0\,(1 + \epsilon\rho_1 + \epsilon^2\rho_2 + \;......),$$

$$T^* = \quad 1 + \epsilon\theta_1 + \epsilon^2\theta_2 + \;......),$$

$$u^* = \epsilon u_1 + \epsilon^2 u_2 + \;......,$$

and similar expressions for v^* and w^*. Here $\rho_0 = \exp\,(\frac{1}{2}\gamma\, M^2 r^2)$, M is the Mach number and the starred variables are non-dimensional. The notation used in Iwao Harada[5] is followed and not repeated here. The axisymmetric flow is considered with respect to a rotating frame with angular velocity $\bar{\Omega}$. Linearising the equations governing mass, momentum and energy and the equation of state for a perfect gas and then using the stretched variables for the boundary layer flow as

$$\rho_1 = \rho,\; u_1 = r^* u,\; v_1 = r^* v,\; w_1 = (E/\rho_0)^{1/2}\, w,\; \theta_1 = \theta,$$

$$\rho_1 + \theta_1 = p_1 = (E/\rho_0)\, p.\; \zeta = (E/\rho_0)^{-1/2} z,\; t = t^*,$$

the following differential equations are obtained representing the flow under boundary layer approximation[5].

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r^2 u) + \frac{\partial w}{\partial \zeta} + \gamma M^2 r^2 u = 0,$$

$$\frac{\partial u}{\partial t} - 2v + \theta = \frac{\partial^2 u}{\partial \zeta^2},$$

$$\frac{\partial v}{\partial t} + 2u = \frac{\partial^2 v}{\partial \zeta^2},$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\gamma M^2}\frac{\partial p}{\partial \zeta} + \frac{4}{3}\frac{\partial^2 w}{\partial \zeta^2},$$

$$\frac{\partial \theta}{\partial t} + (\gamma - 1)\left[\frac{1}{r}\frac{\partial}{\partial r}(r^2 u) + \frac{\partial w}{\partial \zeta}\right] = \frac{\gamma}{P_r}\frac{\partial^2 \theta}{\theta \zeta^2},$$

$$\rho + \theta = 0,$$

where E is the Ekman number, P_r is the Prandtl number, M is the Mach number and γ is the ratio of specific heats. In writing the above equations, the superscripts and the subscripts are dropped. The initial and boundary conditions are

$$\rho = u = v = w = \theta = 0 \ (t = 0),$$

$$u = 0,\ v = \beta \exp(int),\ w = 0,\ \theta = 0 \text{ at } \zeta = 0 \ \ (t > 0),$$

$$u \to 0,\ \lim v,\ \lim \theta \text{ exist as } \zeta \to \infty,$$

where $n = n'/\Omega$. Using the Laplace transform, the differential equations for the transformed variables are

$$s\bar{\rho} + \frac{1}{r}\frac{\partial}{\partial r}(r^2 \bar{u}) + \frac{\partial \bar{w}}{\partial \zeta} + \gamma M^2 r^2 \bar{u} = 0 \tag{1}$$

$$L_\zeta \bar{u} = \bar{\theta} = 2\bar{v} \tag{2}$$

$$L_\zeta \bar{v} = 2\bar{u} \tag{3}$$

$$\left(\frac{4}{3} L_\zeta + \frac{1}{3} s\right)\bar{w} = \frac{1}{\gamma M^2}\frac{\partial \bar{\bar{p}}}{\partial \zeta} \tag{4}$$

$$[L_\zeta + (1 - Pr)s]\,\bar{\theta} = -4(\delta^2 - 1)\ \bar{u} \tag{5}$$

$$\bar{\rho} + \bar{\theta} = 0 \tag{6}$$

where

$$L_\zeta = \frac{\partial^2}{\partial \zeta^2} - s,\ \delta^2 = 1 + \frac{1}{4}(\gamma - 1)\,Pr\,M^2 r^2.$$

The Laplace transformed boundary conditions are

$$\bar{u} = 0,\ \bar{v} = \beta/(s - in),\ \bar{w} = 0,\ \bar{\theta} = 0 \text{ at } \zeta = 0 \tag{7}$$

$$\bar{u} \to 0,\ \bar{v},\ \theta \to \text{finite values as } \zeta \to \infty. \tag{8}$$

3. Solution

It is difficult to solve the Eqns. (1) to (6) subject to the boundary conditions (7) and (8) in the case $Pr \neq 1$. However, the effects of compressibility on the flow system can be still studied by assuming the Prandtl number as unity. Further, no terms will be lost by this assumption in the steady solution in the non-oscillatory case. The solution for the transformed variables when $Pr = 1$ is given by

$$\bar{\theta} = \frac{2\beta(\delta^2-1)}{(s-in)\,\delta^2}\left[h_0 - \frac{1}{2}(h_1+h_2)\right],$$

$$\bar{\rho} = -\bar{\theta}$$

$$\bar{u} = \frac{i\beta}{2\delta(s-in)}(h_1-h_2),$$

$$\bar{v} = \frac{\beta}{\delta^2(s-in)}\left[(\delta^2-1)h_0 + \frac{1}{2}(h_1+h_2)\right],$$

$$\bar{w} = \int_0^{\zeta}\left[s\bar{\theta} - \frac{1}{r}\frac{\partial}{\partial r}(r^2\bar{u}) - \gamma M^2 r^2 \bar{u}\right]d\zeta,$$

where

$$h_0 = \exp(-\zeta\sqrt{s}),\; h_{1,2} = \exp(-\zeta\sqrt{s \pm 2i\delta}).$$

Inverting these functions, the solution for the original variables is given by

$$u = \frac{i\beta}{4\delta}(f_1-f_2)\exp(int) \tag{9}$$

$$v = \frac{\beta(\delta^2-1)}{2\delta^2}\left[(\delta^2-1)f_0 + \frac{1}{2}(f_1+f_2)\right]\exp(int) \tag{10}$$

$$\theta = \frac{\beta(\delta^2-1)}{\delta^2}\left[f_0 - \frac{1}{2}(f_1+f_2)\right]\exp(int) \tag{11}$$

$$\begin{aligned}
w = {} & \frac{2\beta(\delta^2-1)}{\delta^2(\pi t)^{1/2}}\left[1-\exp(-\zeta^2/4t)\right]\Bigg[1-\cos 2\delta t \\
& - \frac{it}{\zeta}\left(\frac{1}{n_1}\exp(-2i\delta t) + \frac{1}{n_2}\exp(2i\delta t)\right)\Bigg] \\
& + \frac{2\beta(\delta^2-1)(in)^{1/2}}{\delta^2}\left[\frac{1}{2}g_0 + \operatorname{erf}\sqrt{int}\right]\exp(int) \\
& + \sum_{j=1}^{2} A_j\Bigg[(in_1)^{-1/2}\left(\frac{1}{2}g_1 + \operatorname{erf}\sqrt{in_1 t}\right) - B_j(in_2)^{-1/2}\Bigg(\frac{1}{2}g_2 \\
& + \operatorname{erf}\sqrt{in_2 t}\Bigg)\Bigg]\exp(int) - \frac{i\beta(\delta^2-1)\zeta}{4\delta^2}\left[\frac{f_1}{n_1}+\frac{f_2}{n_2}\right]\exp(int) \\
& - \frac{i\beta\gamma M^2 r^2\zeta}{8\delta}[f_1-f_2]\exp(int) - \frac{\beta(\delta^2-1)}{2\delta^2}
\end{aligned}$$

$$\left[(in_1)^{-3/2}\left(\frac{1}{2}\ g_1 + \text{erf}\sqrt{in_1 t}\right) + (in_2)^{-3/2}\left(\frac{1}{2}\ g_2 + \text{erf}\sqrt{in_2 t}\right)\right] \exp(int) \tag{12}$$

where

$$n_0 = n,\ n_1 = n + 2\delta,\ n_2 = n - 2\delta,$$

$$A_1 = \frac{-i\beta}{\delta}\left[1 + A_2\ \frac{-i\beta n(\delta^2-1)}{\delta^2},\ \beta_1 = -\beta_2 - 1\ \frac{(\delta^2-1)}{2\delta^2} + \frac{\gamma M^2 r^2}{4}\right],$$

$$f_i = \exp(\zeta\sqrt{in_i})\ \text{erfc}\left(\frac{\zeta}{2\sqrt{t}} + \sqrt{in_1 t}\right) + \exp(-\zeta\sqrt{in_i})\ \text{erfc}\left(\frac{\zeta}{2\sqrt{t}} - \sqrt{in_i t}\right) \tag{13}$$

$$g_i = \exp(\zeta\sqrt{in_i})\ \text{erfc}\left(\frac{\zeta}{2\sqrt{t}} + \sqrt{in_i t}\right) - \exp(-\zeta\sqrt{in_i})\ \text{erfc}\left(\frac{\zeta}{2\sqrt{t}} - \sqrt{in_i t}\right) \tag{14}$$

The steady state solution is obtained by taking the limit $t \to \infty$.

$$u = \frac{i\beta\exp(int)}{2\delta}\ [\exp(-\zeta\sqrt{in_1}) - \exp(-\zeta\sqrt{in_2})] \tag{15}$$

$$v = \frac{\beta\exp(int)}{\delta^2}\left[(\delta^2-1)\exp(-\zeta\sqrt{in}) + \frac{1}{2}\exp(-\zeta\sqrt{in_1}) + \frac{1}{2}\exp(-\zeta\sqrt{in})_2\right] \tag{16}$$

$$\theta = \frac{2\beta(\delta^2-1)}{\delta^2}\exp(int)\left[\exp(-\zeta\sqrt{in}) - \frac{1}{2}\exp(-\zeta\sqrt{in_1}) + \frac{1}{2}\exp(-\zeta\sqrt{in_2})\right] \tag{17}$$

$$w = \left\{\frac{-2\beta(\delta^2-1)\sqrt{in}}{\delta^2}\left[\exp(-\zeta\sqrt{in}) - 1\right] - \sum_{j=1}^{2} A_j\left[(in_1)^{-1/2}\right.\right.$$

$$\left.\exp(-\zeta\sqrt{in_1}) - (in_2)^{-1/2}\exp(-\zeta\sqrt{in_2}) - (in_1)^{-1/2} + B_j(in_2)^{-1/2}\right]$$

$$-\ \frac{i\beta\zeta(\delta^2-1)}{2\delta^2}\left[\frac{\exp(-\zeta\sqrt{in_1})}{n_1} + \frac{\exp(-\zeta\sqrt{in_2})}{n_2}\right]$$

$$-\ \frac{i\beta\gamma M^2 r^2\zeta}{4\delta}\left[\exp(-\zeta\sqrt{in_1}) - \exp(-\zeta\sqrt{in_2})\right]$$

$$- \frac{\beta(\delta^2 - 1)}{2\delta^2} \left[\frac{\exp(-\zeta\sqrt{in_1})}{(in_1)^{3/2}} + \frac{\exp(-\zeta\sqrt{in_2})}{(in_2)^{3/2}} \right.$$

$$\left. - (in_1)^{-3/2} - (in_2)^{-3/2} \right] \Big\} \exp(int) \tag{18}$$

4. Discussion and Results

Wave motion in the boundary layers

The solution represented by Eqns. (9) - (12) gives a unified representation of the initial Rayleigh flow, the inertial oscillations and Ekman-Stokes layers on the disk. The presence of the error functions shows that the spin-up process admits generation and propagation of diffused waves. To demonstrate this we make use of a result due to Strand (6) on the representation of complementary error function wtih complex argument. Strand (6) has shown that for $x > 0$, $y \geqslant 0$,

$\operatorname{erfc}(x + iy) = \phi(x,y) \exp(-2ixy)$

where

$$\phi(x,y) = \sum_{m=0}^{\infty} (xy)^{2m} \left[\gamma_m(x) - ixy(m+1)\gamma_{m+1}(x) \right],$$

$$\gamma_{m+1}(x) = \frac{2}{(2m+1)\sqrt{\pi}} \left[\frac{\exp(-x^2)}{(m+1)!\,x^{2m+1}} - \frac{\sqrt{\pi}\gamma_m(x)}{(m+1)} \right], \; m = 0,1,2...$$

$$\gamma_0(x) = \operatorname{erfc} x.$$

Since $\operatorname{erf}(-x - iy) = -\operatorname{erf}(x + iy)$ and $\operatorname{erf}(x - iy) = \overline{\operatorname{erf}(x + iy)}$, these cases are also covered by the above expressions, but the case $x = 0$ is not covered. $\phi(x, y)$ is a complex function and $\phi(x.y) \to 0$ as $x \to \infty$. Using the Strand series, the functions f_i and g_i can be expressed as follows. For convenienee of interpretting the boundary layer thickness and to study the relative roles of various effects on the flow, the origional dimensional variables are used. If $E^* = (\Omega E/\rho_0)^{1/2}$ and $C = \sqrt{2\Omega n_1} E^* L$, we have

Case (i) $z' > Ct'$

$$f_1 \exp(int) = \exp(-2i\delta\Omega t')$$

$$\cdot \left\{ \bar{\phi}\left(\frac{z' - Ct'}{2E^*L\sqrt{t'}}, \sqrt{\Omega n_1 t'/2} \right) \exp\left(-\sqrt{\Omega n_1}\; z'/\sqrt{2E^*L} \right) \right.$$

$$\left. + \phi\left(\frac{z' + Ct'}{2E^*L\sqrt{t'}}, \sqrt{\Omega n_1 t'/2} \right) \exp\left(\sqrt{\Omega n_1}\; z'/\sqrt{2E^*L} \right) \right\} \tag{19}$$

Case (ii) $z' < Ct'$

$$f_1 \exp(int) = \exp(-2i\delta\Omega t')$$
$$\left\{-\phi\left(\frac{|z' - Ct'|}{2E^*L\sqrt{t'}}, \sqrt{\Omega n_1 t'/2}\right)\exp\left(-\sqrt{\Omega n_1}z'/\sqrt{2}E^*L\right)\right.$$
$$\left.+\phi\left(\frac{z' + Ct'}{2E^*L\sqrt{t'}}, \sqrt{\Omega n_1 t'/2}\right)\exp\left(\sqrt{\Omega n_1}z'/\sqrt{2E^*L}\right)\right\}$$
$$+ 2\exp[-(1+i)z'\sqrt{\Omega n_1}/\sqrt{2E^*L}] \tag{20}$$

where the superscript(′) denotes dimensional quantity. The other functions f_i and g_i appearing in the solution can also be expanded using the Strand series and the lengthy expressions are not presented to save space.

Expressed in this from, the function f_1 clearly shows the wave like behaviour and expresses vorticity as diffusion from a source travelling with velocity $\sqrt{2(n' + 2\delta\Omega)}$ E^*L away from the disk. Equation (19) represents the flow ahead of the wave front and Eqn. (20) represents the flow behind the wave front. The effective decay length of the wave is $\sqrt{2E^*L}/\sqrt{n' + 2\delta\Omega}$ so that the wave damps out in a time of order $(n' + 2\delta\Omega)^{-1}$. This attenuation of the wave system is a consequence of the interaction of the diffusing shear layer with inertial oscillations, Coriolis force and compressibility. The second term in Eqn. (19) or Eqn. (20) is the mirror image of the travelling source. The last term in Eqn. (20) leads to the steady Ekman-Stokes state.

The terms in Eqn. (12) containing f_1, f_2, g_1, g_2 correspond to propagation of diffusing waves with velocities $\sqrt{2\,|\,n' \pm 2\delta\Omega\,|}\; E^*L$ and penetrate through distances of order $\sqrt{2E^*L}\,\sqrt{|\,n' \pm 2\delta\Omega\,|}$. The distance of penetration of these waves reduce to a modified Stokes layer of thickness order $\sqrt{2E^*L}/\sqrt{|\,n' \pm 2\Omega\,|}$ in the case of an incompressible fluid ($\delta = 1$). This, in turn, will be a Stokes layer of thickness order $(\nu/n')^{1/2}$ if $n' >> \Omega$ and the viscous force balances the acceleration. It will be an Ekman layer when $n' << \Omega$ and the viscous force would balance the Coriolis force.

The term in Eqn. (18) containing g_0, namely $\beta\sqrt{in(\delta^2 - 1)}\delta^{-2}\exp(int)$ arises due to the presence of the forcing frequency n of the disk. When expanded using Strand series, this shows the existence of diffused wave motion with phase velocity $\sqrt{2n'}\; E^*L$ and the wave decays over a distance of order $\sqrt{2E^*L}/\sqrt{n'}$. Thus the forcing frequency of the disk not only influences the wave motion existing in the non-oscillatory case but also induces new waves in addition to those observed in the non-oscillatory case.

In the case of low frequency oscillations $n' << 2\delta\Omega$, the waves represented by the term containing g_0 travel slower than the other waves occuring in the solution given by Eqn. (12). The depths of penetration of various wave systems are of order

$$\sqrt{2E^*L}/\sqrt{n'}\,,\ \sqrt{2E^*L}/\sqrt{|\,n' \pm 2\delta\Omega\,|}\quad (<\sqrt{2E^*L}/\sqrt{n'})$$

so that the depth of penetration, in effect, is of order $\sqrt{2E^*L}\,\sqrt{n'}$. It is of interest to note that even small values of the forcing frequency alters the depth of penetration of the waves observed in the non-oscillatory case.

For high frequency oscillations $n' >> 2\delta\Omega$, the velocities of propagation of various wave modes is of order $\sqrt{2n'}\ E^*L$ and the decay length is of order $\sqrt{2E^*L}/\sqrt{n'}$. In this case also the term containing g_0 in Eqn. (12) plays a significant role in determining the decay length. Further the amplitude of the waves given by this term is of order $\sqrt{n'/\Omega}\,(\delta^2 - 1)\delta^{-2}\beta$ whereas the waves given by other terms in Eqn. (12) have amplitudes of order $(\Omega/n')^{\alpha}\,(\delta^2 - 1)\,\delta^{-2}\beta : \alpha = \frac{1}{2}, 1, \frac{3}{2}$. Consequently the waves induced by the forcing frequency n and represented by the term containing g_0 have larger amplitudes in comparison with other waves when n is large. The effects of compressibility and rotation become relatively unimportant when high frequency oscillations are imposed on the disk.

The case $|\, n \pm 2\delta \,| << 1$ is of special interest. The decay length tends to be large as $n \to \pm 2\delta$ ($n' \to \pm 2\delta\Omega$). The frequencies $n' = \pm 2\delta\Omega$ are resonant frequencies, which reduce to $n' = \pm 2\Omega$ for an incompressible fluid.

The first term in Eqn. (12) shows that the fluid region also supports stationary waves and the fluid region diffuses parabolically with a characteristic diffusion constant $2E\Omega\rho_0^{-1} L^2$. The other terms in Eqn. (12) containing erf $\sqrt{i(n' + 2\delta\Omega)t'}$ and erf $\sqrt{in't'}$ lead to final steady Ekman-Stokes state for a compressible gas.

Steady motion in the boundary layer and the flow outside

The ultimate steady state solution given by Eqns. (15)-(18) shows the existence of multiple boundary layers on the disk, of thickness orders

$$\sqrt{2E^*L}/\sqrt{n'},\ \sqrt{2E^*L}/\sqrt{|\, n' + 2\delta\Omega \,|},\ \sqrt{2E^*L}/\sqrt{|\, n' - 2\delta\Omega \,|}\ .$$

The first mentioned layer is absent in the non-oscillatory case ($n' = 0$). For low frequency oscillations the boundary layer is effectively a Stokes layer of thickness order $E^*L\,\sqrt{2/n'}$ since the later two layers manifesting mainly through Coriolis force and compressibility effects are of smaller thickness. In the case of high frequency oscillations the boundary layers will all penetrate to a depth of same order. Thus for both low and high frequency oscillations, it is the Stokes layer of thickness order $E^*L\sqrt{2/n'}$ that is effectively formed on the disk and it is only for the intermediate frequencies that the compressibility plays a major role in determining the boundary layer thickness. When the exponentials in the steady solution decay, the remaining terms give an interior flow independent of ζ. In the non-oscillatory case this interior flow in w corresponds to a secondary flow similar to that arising in the spin-up problems for incompressible fluids. The value of w as $\zeta \to \infty$ shall estimate the suction velocity at infinity, but for the non-linear corrections.

The azimuthal velocity v tends to zero as ζ tends to infinity in the non-oscillatory case of an incompressible fluid. The compressibility of the fluid brings into effect (when $n = 0$) a non-zero azimuthal velocity at the edge of the boundary layer. It is of interest to note that even for a compressible gas the presence of the forcing frequency of oscillation of the disk will again reduce the azimuthal velocity at the edge of the boundary layer to zero. The same is true with the temperature also.

5. Conclusion

It is established that the oscillations of the disk produce additional waves in comparison with the non-oscillatory case. These waves penetrate through longer distances in the fluid when the frequency is small. The compressible Stokes layers on the disk observed in the non-oscillatory case are modified by the frequency of oscillation. Further, a new layer exists due to the frequency of oscillation. Thus the frequency of oscillation plays a major role in determining the thickness of the compressible Stokes layer and is a mechanism of controlling the thickness of the boundary layer on the disk. Thus the theory is useful in defence oriented fluid dynamic problems like the one of gas centrifuges.

References

1. Howard, L. N.. Moore, D. W., & Spiegel, E. A., *Nature*. **214**. (1967), 1297.
2. Jones, A. F., *J. Fluid Mech.*, **39** (1969), 257.
3. Claire Jacobs., *Q. Jl. Mech. appl. Math.*, **24** (1971), 221.
4. Venkatasiva Murthy, K. N., *Appl. Sci. Res.* **35** (1979). 111.
5. Iwao Harada. *J. Phys. Soc. Jpn.*. **47** (1979), 1313.
6. Strand, O. N., *Math. Comp.*, **19** (1965), 127.