INFORMATION GENERATING FUNCTION FOR THE r-POWER DISTRIBUTION

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In a previous paper the authors defined an information generating function for giving Kapur's entropy of order α and type β for a given distribution P. In the present paper they deduce the corresponding generating function for the r-power distribution $P^{(r)}$ obtained from P. The generating function is specialized to the cases of Rényi's and Shannon's entropies. Entropy of $P^{(r)}$ is also obtained for some special distributions P.

Information generating functions are useful in obtaining the entropy for a given probability distribution. However, if the generating function is known, it can be inverted uniquely to give the corresponding probability distribution. Golomb[1] and Sharma[2] have worked on information generating functions which give respectively Shannon's[3] and Rényi's[4] entropy. In a previous paper Mathur & Kashyap[5], we gave a generalised information generating function from which Kapur's entropy[6] of order α and type β can be found out for a given distribution P. In the present paper we give a further generalisation of the generating function which gives Kapur's entropy for the r-power[7] distribution $P^{(r)}$ when the distribution P is known. The generating function is specialised for the cases of Rényi's and Shannon's entropies. We append some examples in which the entropies of the distribution $P^{(r)}$ are calculated corresponding to the cases when P is one of the standard distributions, e.g. the uniform, the geometric, the exponential, the Gamma and the normal distributions. Finally, it is observed that the r-power distributions corresponding to the Gamma and the normal distributions are themselves Gamma and normal distributions respectively with their parameters modified. The same result holds for the exponential distribution as a particular case of the Gamma distribution.

GENERALISED GENERATING FUNCTION

Let

$$P = \left\{ p_i \right\}_{i=1}^{N}$$

be a probability distribution complete or incomplete, and

$$P^{(r)} = \left\{ p_i' \right\}_{i=1}^{N}$$

be the r-power distribution obtained from it where

$$p_i' = p_i^r / \Sigma p_i^r$$

The authors[5] have earlier defined the generalised information generating function for P as

$$\hat{M}_{\alpha,\beta}(P\,;u)=\left(\frac{\Sigma p_i^{\alpha+\beta-1}}{\Sigma p_i^{\beta}}\right)^{\frac{u}{1-\alpha}},\ \alpha\neq 1\,,\ \beta>0\,,\ \alpha+\beta=1>0 \tag{1}$$

from which we obtained

$$\partial/\partial u\, M_{\alpha,\beta}(P\,;u)\Big|_{u=0}=H_{\alpha}^{\beta}(P)=\frac{1}{1-\alpha}\log\frac{\Sigma p_i^{\alpha+\beta-1}}{\Sigma p_i^{\beta}} \tag{2}$$

where $H_{\alpha}^{\beta}(P)$ is Kapur's entropy of order α and type β. In the above and subsequently 'Σ' stands for summation over $i=1$ to N and natural logarithms are used, so that the entropies are in natural units.

From (1) we have the generating function for the distribution $P^{(r)}$ as

$$M_{\alpha,\beta}(P^{(r)};u)=(\Sigma p_i^{r})^{u}\left(\frac{\Sigma p_i^{r(\alpha+\beta-1)}}{\Sigma p_i^{r\beta}}\right)^{u/(1-\alpha)} \tag{3}$$

If the value of r satisfies the inequality

$$\Sigma p_i^{r}\leqq 1,\ \text{then}\ P^{r}=\left\{p_i^{r}\right\}_{i=1}^{N}$$

a probability distribution and we may write

$$M_{\alpha,\beta}(P^{(r)};u)=(\Sigma p_i^{r})^{u}\,M_{\alpha,\beta}(P^{r},u)\,, \tag{4}$$

whence using (2) we have for the entropy of $P^{(r)}$

$$H_{\alpha}^{\beta}(P^{(r)})=H_{\alpha}^{\beta}(P^{r})+\log\Sigma p_i^{r}$$

or

$$H_{\alpha}^{\beta}(P^{(r)})=H_{\alpha}^{\beta}(P^{r})-H_{\alpha}^{\beta}(\Sigma p_i^{r}) \tag{5}$$

Remark 1: It may be observed that (5) is a generalisation of a similar result of Kapur[3] concerning the generalised Shannon entropy, viz.

$$H_1\left(\frac{p_1}{w(P)},\frac{p_2}{w(P)},\ldots,\frac{p_N}{w(P)}\right)=H_1(p_1,p_2,\ldots,p_N)-H_1(\Sigma p_i) \tag{6}$$

Remark 2: In case P is a continuous distribution the above equations (1) through (5) may be modified by changing p_i to the density function $p(x)$ and summation to integration with respect to x. In this case the modified form of (5) will give a generalisation of the continuous analogue of Kapur's result.

Examples

Uniform Distribution: Here we have

$$p_i=\frac{1}{N}\ \forall\ i=1,2,\ldots,N.$$

Hence it can be seen that

$$M_{\alpha,\beta}(P^{(r)};u)=M_{\alpha,\beta}(P;u)=N^{u}\,,$$

which is independent of α and β. So in this case the entropy $H_\alpha^\beta(P)$ is independent of α and β. Also the r-power distribution $P^{(r)}$ will be the same as P for all r.

Geometric Distribution : Here

$$p_i = qp^i, \; p + q = 1, \; i = 0, 1, 2, \ldots,$$

$$M_{\alpha,\beta}(P; u) = \left(\frac{q^{\alpha-1}(1-p^\beta)}{1-p^{\alpha+\beta-1}} \right)^{u/(1-\alpha)}$$

$$M_{\alpha,\beta}(P^{(r)}; u) = \left(\frac{q^r}{1-p^r} \right)^u \left(\frac{q^{r(\alpha-1)}(1-p^{r\beta})}{1-p^{r(\alpha+\beta-1)}} \right)^{u/(1-\alpha)}$$

Exponential Distribution : Here

$$p(x) = \lambda e^{-\lambda x}, \; \lambda > 0, \; x \in [0, \infty]$$

$$M_{\alpha,\beta}(P; u) = \left(\frac{\lambda^{\alpha-1}\beta}{\alpha+\beta-1} \right)^{u/(1-\alpha)}$$

$$M_{\alpha,\beta}(P^{(r)}; u) = \left(\frac{(\lambda r)^{\alpha-1}\beta}{\alpha+\beta-1} \right)^{u/(1-\alpha)}$$

We now proceed to consider the particular simpler cases of the above.

GENERATING FUNCTIONS FOR RENYI'S ENTROPY

(*a*) For an incomplete distribution, setting $\beta = 1$ in (1), (4), (5), we have

$$M_\alpha(P; u) = M_{\alpha,1}(P; u) = \left(\frac{\Sigma p_i^\alpha}{\Sigma p_i} \right)^{u/(1-\alpha)} \quad (7)$$

$$M_{\alpha,1}(P^{(r)}; u) = (\Sigma p_i^r)^u \, M_{\alpha,1}(P^r; u) \quad (8)$$

$$H_\alpha(P^{(r)}) = H_\alpha(P^r) - H_\alpha(\Sigma p_i^r) \quad (9)$$

(*b*) In case P is a complete distribution, (7) becomes

$$M_\alpha(P; u) = (\Sigma p_i^\alpha)^{u/(1-\alpha)}, \quad (10)$$

which is the generating function given by Sharma[2].

Using (10), (8) and (9) can be written

$$M_\alpha(P^{(r)}; u) = \frac{[M_{\alpha r}(P; u)]^{(1-\alpha r)/(1-\alpha)}}{[M_r(P; u)]^{\alpha(1-r)/(1-\alpha)}} \quad (11)$$

$$H_\alpha(P^{(r)}) = \frac{1-\alpha_r}{1-\alpha} H_{\alpha r}(P) - \frac{\alpha(1-r)}{1-\alpha} H_r(P). \quad (12)$$

GENERATING FUNCTIONS FOR SHANNON'S ENTROPY

Incomplete Distribution

Letting $\alpha \to 1$ in (7), (8) and (9), we have

$$M(P;u) = \lim_{\alpha \to 1} M_\alpha(P;u) = \exp\left[-u \frac{\Sigma p_i \log p_i}{\Sigma p_i}\right] \tag{13}$$

$$M(P^{(r)};u) = (\Sigma p_i^{\,r})^u \, M(P^r;u) \tag{14}$$

$$H(P^{(r)}) = H(P^r) - H(\Sigma p_i^{\,r}) \tag{15}$$

Complete Distribution

Equations (13) to (15) can be specialised for the case when P is is a complete distribution as before. However in this case the generating function given by Golomb[1], viz.

$$T(u) = \Sigma p_i^{\,u} \tag{16}$$

is much simpler, from which

$$-\frac{d}{du} T(u) \Big|_{u=1} = -\Sigma p_i \log p_i = H(P) \,, \tag{17}$$

where $H(P)$ is Shannon's entropy. From (16) and (17) we have the generating function and entropy for the r-power distribution $P^{(r)}$ as

$$T_r(u) = \frac{T(ru)}{[T(r)]^u} \tag{18}$$

$$H(P^{(r)}) = \log T(r) - r \frac{d}{dr} \log T(r) \tag{19}$$

Examples

We will now give the generating function discussed under Complete Distribution and the entropy for the r-power distributions corresponding to some standard distributions.

Gamma Distribution : Here we have

$$p(x) = \frac{\lambda^k}{\Gamma(k)} e^{-\lambda x} x^{k-1} \,, \lambda > 0, x \epsilon [0, \infty]$$

Hence

$$T(u) = \frac{\lambda^{u-1} u^{u-uk-1} \Gamma(uk-u+1)}{[\Gamma(k)]^u}$$

$$H(P) = -\frac{d}{du} T(u)\Big|_{u=1} = k - \log\lambda + \log\Gamma(k) - (k-1)\Psi(k)$$

where the function $\Psi(k)$ is defined by Erdélyi[9]

$$\Psi(k) = \lim_{n\to\infty}\left[\log n - \frac{1}{k} - \frac{1}{k+1} - \frac{1}{k+2} - \cdots - \frac{1}{k+n}\right]$$

$$T_r(u) = \frac{u^{ur-urk-1} (\lambda r)^{u-1} \Gamma(ukr-ur+1)}{[\Gamma(rk-r+1)]^u}$$

$$H(P^{(r)}) = -\frac{d}{du} T_r(u)\Big|_{u=1} = \log\left[\frac{\lambda^{r-1} \quad r^{r-rk-1} \quad \Gamma(rk-r+1)}{[\Gamma(k)]^r}\right]$$

$$-r\left(\log\lambda + (1-k)\log r + \frac{r-rk-1}{r} - \log\Gamma(k)\right) + r(k-1)\,\Psi(rk-r+1)$$

Exponential Distribution : For $k = 1$, Gamma distribution reduces to exponential distribution; therefore the corresponding $T(u)$, $T_r(u)$ and entropies can be obtained from Gamma distribution as a particular case.

Normal Distribution : Here

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-x^2/2\sigma^2}, \quad \sigma > 0, \ x \in (-\infty, \infty)$$

$$T(u) = u^{-1/2}\left(2\pi\sigma^2\right)^{(1-u)/2}$$

$$T_r(u) = u^{-1/2}\left(\frac{2\pi\sigma^2}{r}\right)^{(1-u)/2}$$

$$H(P^{(r)}) = -\frac{d}{du} T_r(u)\Big|_{u=1} = 1/2 = 1/2\log r + 1/2\log(2\pi\sigma^2)$$

From a comparison of $T(u)$ and $T_r(u)$ obtained above we observe :

Remark 1 : The r-power distribution corresponding to the Gamma distribution is another Gamma distribution with the parameters k and λ replaced by $kr - r + 1$ and λr respectively. Considering the case $k = 1$, we see that the r-power distribution of the exponential distribution is exponential with λ replaced by λr.

Remark 2 : The r-power distribution corresponding to the normal distribution with variance σ^2 is a normal distribution with variance σ^2/r.

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REFERENCES

1. GOLOMB, S. W., *IEEE Trans. Inf. Theory*, **IT-12** (1966), 75.
2. SHARMA, B. D., On Generating Function of Rényi's Entropy, *Internat. Conf. Operational Research in Universities*, Delhi (1967).
3. SHANNON, C. E., *Bell Syst. Tech. J.*, **27** (1948), 379.

4. RENYI, A., *4th Berkeley Symp. Prob. Stat.*, I (1961), 547.

5. MATHUR, K. C. & KASHYAP, B. R. K., A Generalised Information Generating Function, (communicated for publication), 1969.

6. KAPUR, J. N., *Math. Seminar* (Delhi), 4 (1967).

7. KASHYAP, B. R. K., Some Results on Entropy in Information Theory, *Cahiers du Centre d'Etudes de Recherche Oprationnelle* (Bruxelles), 12 (1970).

8. KAPUR, J. N., Presidential address, Section of Maths. The fifty-fifth session of the Indian Science Congress, Varanasi, 1968.

9. ERDELYI, A., "Higher Transcendental Functions", Vol. I (McGraw Hill), 1953.