OPTIMUM TRAJECTORIES WITH SPECIFIED TRANSFER ANGLE

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Inter-orbital optimum trajectories with specified transfer angle in an inverse square gravitational field have been analysed. Criterion of optimization adopted is minimum velocity increment in the entire transfer operation with one impulse each at the initial and final terminal. Particular cases of the above problem are discussed and as a numerical illustration, results are obtained for transfer trajectories between two orbits in Earth's gravitational field.

Srivastava & Singh[1] have studied optimum transfer trajectory between any two terminals constrained by a specified transfer angle in an inverse square gravitational field, the initial terminal being a point on an elliptic orbit and final terminal a point on another elliptic orbit. The optimum trajectory defined there[1] was one which minimizes the single velocity increment applied to the rocket at the initial terminal. In the present paper optimum inter-orbital trajectory with specified transfer angle is analysed under a different optimization criterion which is defined as that which minimizes the total characteristic velocity required in the entire two impulsive transfer operation. Two particular cases of the above problem are : (i) when the mission is to achieve inter-orbital transfer with least velocity increment at the initial terminal which is the problem studied earlier[1], (ii) when the mission is to minimize the velocity increment applied at the final terminal for rocket's entry into the final orbit. Case (ii) has been discussed in some detail. Numerical results are also obtained for orbits in Earth's gravitational field.

FUNDAMENTAL EQUATIONS AND CHARACTERISTIC VELOCITY

Let the equations of the elliptic orbits corresponding to initial and final terminals be

$$l_1 = r\,(1 + e_1 \cos\theta) \tag{1}$$

$$l_2 = r\,[\,1 + e_2 \cos(\theta - \alpha)\,] \tag{2}$$

where suffixes (1) and (2) relate to initial and final terminals, l is semi-latus rectum, e is eccentricity, $(r\,,\,\theta)$ are polar co-ordinates with force centre (focus) as the pole and the line joining the force centre to the peri-apsis of the initial orbit as the initial line and α is the angle between the major axes of the initial and final orbits.

If $(r_1\,,\,\theta_1)$ and $(r_2\,,\,\theta_1+\phi)$ be the initial and final terminals, equations (1) and (2) give

$$\rho = \frac{r_1}{r_2} = \frac{l_1}{l_2}\left(\frac{1 + e_2\cos(\theta_1 + \phi - \alpha)}{1 + e_1 \cos\theta_1}\right) \tag{3}$$

where ϕ is the specified transfer angle. The velocity change $\triangle V_1$, required at the initial terminal in directing the rocket along the transfer trajectory will be given by

$$\triangle V_1 = [\,V_0^2 + V_1^2 - 2V_0V_1\cos(\gamma_0 - \gamma_1)\,]^{\frac{1}{2}} \tag{4}$$

where V_0, V_1 are rocket velocities just before and after the application of impulse and

γ_0 , γ_1 are the corresponding heading angles. We can show that

$$\tan \gamma_0 = \frac{e_1 \sin \theta_1}{1 + e_1 \cos \theta_1} \tag{5}$$

$$V_0^2 = \frac{\mu}{l_1} \left[1 + e_1^2 + 2e_1 \cos \theta_1 \right] \tag{6}$$

where μ is gravitational parameter.

If V_2 is the rocket velocity along the transfer trajectory just before the application of the impulse at the final terminal, V_f is the orbital velocity to be obtained for entry in the final orbit and γ_2 , γ_f the corresponding heading angles, the required velocity change $\triangle V_2$ will be given by the relation,

$$\triangle V_2 = [V_2^2 + V_f^2 - 2 \, V_2 \, V_f \cos (\gamma_2 - \gamma_f)]^{\frac{1}{2}} \tag{7}$$

It can be shown that

$$\tan \gamma_f = \frac{e_2 \sin (\theta_1 + \phi - \alpha)}{1 + e_2 \cos (\theta_1 + \phi - \alpha)} \tag{8}$$

$$V_f^2 = \frac{\mu}{l_2} \left[1 + 2e_2 \cos (\theta_1 + \phi - \alpha) + e_2^2 \right] \tag{9}$$

and

$$V_2^2 = V_1^2 + 2\mu \left(\frac{1}{r_2} - \frac{1}{r_1} \right) \tag{10}$$

By conservation of angular momentum we have

$$\cos \gamma_2 = \frac{V_1 \rho \cos \gamma_1}{V_2} \tag{11}$$

Further, V_1 can be expressed in terms of γ_1 and ϕ as[1]

$$V_1^2 = \frac{\mu \, (1 - \cos \phi) \sec^2 \gamma_1}{r_1 \, (\rho + \sin \phi \tan \gamma_1 - \cos \phi)} \qquad \left| \gamma_1 \right| \leqslant \frac{\pi}{2} \tag{12}$$

Substituting from equations (10) and (11) into equation (7) we have

$$\triangle V_2 = \Bigg[V_1^2 + 2\mu \left(\frac{1}{r_2} - \frac{1}{r_1} \right) + V^2{}_f - 2V_f \left\{ V_1^2 + 2\mu \left(\frac{1}{r_2} - \frac{1}{r_1} \right) \right\}^{\frac{1}{2}} \cos \left\{ \cos^{-1} \left(\frac{V_1 \, \rho \cos \gamma_1}{\left\{ V_1^2 + 2\mu \left(\frac{1}{r_2} - \frac{1}{r_1} \right) \right\}^{\frac{1}{2}}} \right) - \gamma_f \right\} \Bigg]^{\frac{1}{2}} \tag{13}$$

characteristic velocity is then given by

$$\triangle V = | \triangle V_1 | + | \triangle V_2 | \tag{14}$$

where $\triangle V_1$ and $\triangle V_2$ are given by equations (4) and (13). Evidently $\triangle V$ is a function of two variables θ_1 and γ_1.

OPTIMIZATION ANALYSIS

The problem now is to find out values of θ_1 and γ_1 which minimize $\triangle V$. For this we have

$$\frac{\partial (\triangle V)}{\partial \theta_1} = 0 \tag{15}$$

$$\frac{\partial (\triangle V)}{\partial \gamma_1} = 0 \tag{16}$$

From equations (15) and (16) we have

$$\frac{1}{\triangle V_1}\left[V_0 \left\{ \frac{\partial V_0}{\partial \theta_1} - V_1 \sin(\gamma_1 - \gamma_0) \frac{\partial \gamma_0}{\partial \theta_1} - \cos(\gamma_1 - \gamma_0) \frac{\partial V_1}{\partial \theta_1} \right\} + V_1 \left\{ \frac{\partial V_1}{\partial \theta_1} - \cos(\gamma_1 - \gamma_0) \frac{\partial V_0}{\partial \theta_1} \right\} \right] + \frac{1}{\triangle V_2}\left[\xi + V_f \frac{\partial V_f}{\partial \theta_1} - \cos\zeta \left(\xi \eta V_f + \frac{1}{\eta} \frac{\partial V_f}{\partial \theta_1} \right) - V_f \sin\zeta \left[\left\{ 1 - (V_1 \rho \eta \cos\gamma_1)^2 \right\}^{-\frac{1}{2}} \left\{ \left(V_1 \cos\gamma_1 \left(\frac{1}{r_2} \frac{r_1}{\partial \theta_1} - \frac{\rho}{r_2} \frac{\partial r_2}{\partial \theta_1} \right) + \rho \cos\gamma_1 \frac{\partial V_1}{\partial \theta_1} \right) - V_1 \rho \xi \eta^2 \cos\gamma_1 \right\} + \frac{1}{\eta} \frac{\partial \gamma_f}{\partial \theta_1} \right] \right] = 0 \tag{17}$$

and

$$\frac{1}{\triangle V_1}\left[V_0 \left\{ V_1 \sin(\gamma_1 - \gamma_0) - \cos(\gamma_1 - \gamma_0) \frac{\partial V_1}{\partial \gamma_1} \right\} + V_1 \frac{\partial V_1}{\partial \gamma_1} \right] + \frac{1}{\triangle V_2}\left[V_1 \frac{\partial V_1}{\partial \gamma_1} (1 - \eta V_f \cos\zeta) - V_f \sin\zeta \left\{ 1 - (V_1 \rho \eta \cos\gamma_1)^2 \right\}^{-\frac{1}{2}} \left\{ \rho \left(\cos\gamma_1 \frac{\partial V_1}{\partial \gamma_1} - V_1 \sin\gamma_1 \right) - V_1^2 \rho \cos\gamma_1 \eta^2 \frac{\partial V_1}{\partial \gamma_1} \right\} \right] = 0 \tag{18}$$

where

$$\xi = V_1 \frac{\partial V_1}{\partial \theta_1} + \mu \left(\frac{1}{r_1^2} \frac{\partial r_1}{\partial \theta_1} - \frac{1}{r_2^2} \frac{\partial r_2}{\partial \theta_1} \right)$$

$$\eta = \frac{1}{V_2} = \left\{ V_1^2 + 2\mu \left(\frac{1}{r_2} - \frac{1}{r_1} \right) \right\}^{-\frac{1}{2}}$$

$$\zeta = \cos^{-1}(V_1 \rho \eta \cos\gamma_1) - \gamma_f$$

$$\frac{\partial r_1}{\partial \theta_1} = \frac{e_1 l_1 \sin\theta_1}{(1 + e_1 \cos\theta_1)^2}$$

$$\frac{\partial r_2}{\partial \theta_1} = \frac{e_2 l_2 \sin(\theta_1 + \phi - \alpha)}{\{1 + e_2 \cos(\theta_1 + \phi - \alpha)\}^2}$$

$$\frac{\partial V_0}{\partial \theta_1} = -\frac{\mu e_1 \sin\theta_1}{V_0 l_1}$$

$$\frac{\partial \gamma_0}{\partial \theta_1} = \frac{e_1 (e_1 + \cos\theta)}{1 + 2e_1 \cos\theta_1 + e_1^2}$$

$$\frac{\partial V_f}{\partial \theta_1} = -\frac{\mu e_2 \sin(\theta_1 + \phi - \alpha)}{V_f l_2}$$

$$\frac{\partial \gamma_f}{\partial \theta_1} = \frac{e_2 \{ e_2 + \cos (\theta_1 + \phi - \alpha) \}}{1 + 2e_2 \cos (\theta_1 + \phi - \alpha) + e_2^2}$$

$$\frac{\partial V_1}{\partial \gamma_1} = \frac{\mu (1 - \cos \phi)(1 + e_1 \cos \theta_1)}{2 V_1 l_1} \left[\frac{\sec^2 \gamma_1 \ (2A \tan \gamma_1 - \sin \phi \sec^2 \gamma_1)}{A^2} \right]$$

$$\frac{\partial V_1}{\partial \theta_1} = \frac{\mu (1 - \cos \phi) \sec^2 \gamma_1}{2 V_1 l_1} \left[\frac{l_1 (e_1 e_2 \sin (\phi - \alpha) + e_2 \sin (\theta_1 + \phi - \alpha) - e_1 \sin \theta)}{l_2 A^2 (1 + e_1 \cos \theta_1)} - \frac{e_1 \sin \theta_1}{A} \right]$$

and $$A = \frac{l_1}{l_2} \left(\frac{1 + e_2 \cos (\theta_1 + \phi - \alpha)}{1 + e_1 \cos \theta_1} \right) + \sin \phi \tan \gamma_1 - \cos \phi$$

Equations (17) and (18) are two transcendental equations in two unknown θ_1 and γ_1 which can be numerically solved for given values of orbital parameters and ϕ. Having known θ_1 and γ_1, location of initial and final terminals can be obtained from equations (1) and (2) and then optimum values of V_1, V_2 and γ_2 can be evaluated from equations (12), (10) and (11) respectively. γ_0, V_0, γ_f and V_f can be found out from equations (5), (6), (8) and (9) and hence $(\triangle V)_{min}$ can be calculated from equation (14).

PARTICULAR CASES

Two particular cases of the above problem are (i) when the mission is inter-orbital transfer under specified transfer angle constraint with least velocity increment at the initial terminal (ii) when the mission is inter-orbital transfer under specified transfer angle constraint with least velocity increment at the final terminal. In case (i) characteristic velocity will be given by $| \triangle V_1 |$ and equations (17) & (18) are reduced to those obtained by Srivastava & Singh[1]. In case (ii) characteristic velocity is given by $| \triangle V_2 |$ and hence

TABLE 1

PARAMETERS OF OPTIMUM TRANSFER TRAJECTORIES

Mission	Launch heading angle γ_1	Launch vectorial angle θ_1	Launch radius vector r_1 (km)	Arrival radius vector r_2 (km)	Launch velocity V_1 km/sec	Arrival velocity V_2 km/sec	Arrival heading angle γ_2	Velocity change km/sec
General Case	54·05°	123·5°	11985	28555	6·7937	2·4466	47·72°	5·6618*
Case (i)	49.20°	111·0°	11204	26147	7·4203	3·7909	56·76°	$\| \triangle V_1 \|$ *min* = 3·9280
Case (ii)	58·70°	153·5°	13670	32895	6·0178	1·4534	26·62°	$\| \triangle V_2 \|$ *min* = 1·4464

*$(\triangle V)$ *min* $= (| \triangle V_1 | + | \triangle V_2 |)$ *min* $= (4·0519 + 1·6099) = 5·6618$

equations (17) and (18) give

$$\xi + V_f \frac{\partial V_f}{\partial \theta_1} - \cos\zeta \left(\xi\eta V_f + \frac{1}{\eta}\frac{\partial V_f}{\partial \theta_1}\right) - V_f \sin\zeta \left[\left\{ 1 - (V_1\rho\eta \cos\gamma_1)^2 \right\}^{-\frac{1}{2}} \right.$$

$$\left\{ \left(V_1 \cos\gamma_1 \left(\frac{1}{r_2}\frac{\partial r_1}{\partial \theta_1} - \frac{\rho}{r_2}\frac{\partial r_2}{\partial \theta_1}\right) + \rho\cos\gamma_1 \frac{\partial V_1}{\partial \theta_1}\right) - V_1 \rho\xi\eta^2 \cos\gamma_1 \right\}$$

$$\left. + \frac{1}{\eta}\frac{\partial \gamma_f}{\partial \theta_1} \right] = 0 \qquad (19)$$

and

$$V_1 \frac{\partial V_1}{\partial \gamma_1}\left(1 - \eta V_f \cos\zeta \right) - V_f \sin\zeta \left[\left\{ 1 - (V_1\rho\eta\cos\gamma_1)^2 \right\}^{-\frac{1}{2}} \right.$$

$$\left. \left\{ \rho \left(\cos\gamma_1 \frac{\partial V_1}{\partial \gamma_1} - V_1 \sin\gamma_1 \right) - V_1^2 \rho \cos\gamma_1 \eta^2 \frac{\partial V_1}{\partial \gamma_1} \right\} \right] = 0 \quad (20)$$

Equations (19) and (20) can be solved for the unknowns θ_1 and γ_1 for given values of orbital parameters and ϕ and proceeding as explained in general case we can evaluate the optimum values of V_1, V_2, γ_1, γ_2, and $|\triangle V_2|_{min}$

NUMERICAL EXAMPLE

Parameters of the optimum transfer trajectories for the general and particular cases have been numerically obtained and are given in Table 1 for the following two orbits in Earth's gravitational field.

Initial Orbit : $l_1 = 10^4$ km, $e_1 = 0{\cdot}3$

Final Orbit : $l_2 = (2 \times 10^4)$ km, $e_2 = 0{\cdot}4$, $\alpha = 20°$

Specified transfer angle $\phi = 35°$

As should be expected, Table 1 indicates that the velocity changes at the initial and final terminals in the general case are respectively greater than the velocity change at the initial terminal in case (i) and velocity change at the final terminal in case (ii). Fig. 1 and 2 show respectively the variation of $\triangle V$ and $|\triangle V_2|$ with respect to θ_1 for typical values of γ_1. Their study shows that $(\triangle V)_{min}$ and $|\triangle V_2|_{min}$ do occur at $\theta_1 = 123{\cdot}5°$, $\gamma_1 = 54{\cdot}05°$ and $\theta_1 = 153{\cdot}5°$, $\gamma_1 = 58{\cdot}7°$ respectively.

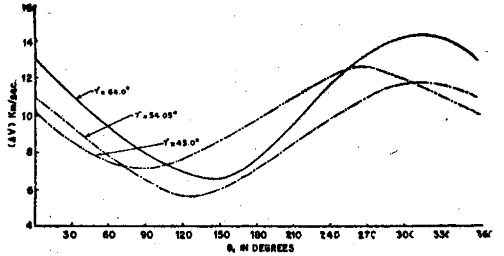

Fig. 1—Variation of $\triangle V$ with respect to vectorial angle.

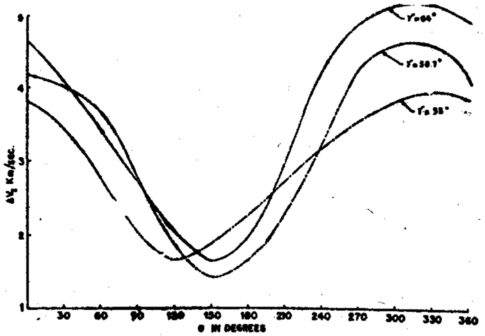

Fig. 2—Variation of $(\triangle V_2)$ with respect to vectorial angle.

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