TWO-DIMENSIONAL BOUNDARY LAYER GROWTH WITH SUCTION

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The boundary layer equations for the unsteady fluid flow with constant suction velocity have been worked out for the impulsive motion of a circular cylinder in the form $V(t) = A \exp(Ct)$ where A and C are certain constants. The stream function has been expanded in terms of some functions $\varkappa_0(s)$ and $\varkappa_1(s)$ where s is a function of y coordinate. The phase angles for various terms have been calculated, and variations shown graphically for large and small frequency of oscillations, where the oscillatory motion is obtained on replacing C by $i\omega$.

The problem of the boundary, layer growth on a circular cylinder which starts its motion from the rest by an impulsive force was taken up by Blasius[1] who also considered time of separation for the second approximation. Goldstein & Rosenhead[2] calculated the time of separation for the above cited problem upto third approximation. Later on Watson[3] considered the boundary layer growth for the unsteady two dimensional flow and solutions for purpose of similarity were also discussed. Lal[4] extended the application of asymptotic boundary layer theory of Watson to the flow past a porous sphere. Nanda[5] considered the boundary layer growth with constant suction when the velocity of the cylinder varies with time, t, in the following two ways :

(i) $V(t) = At^n$

(ii) $V(t) = A \exp(Ct), C > 0$

He calculated the velocity components u, and v and the time of separation. Earlier Lal[6] has considered the growth of boundary layer when the velocity of suction is constant and the velocity of cylinder is of the form $V(t) = A \exp(Ct)$, but the motion is oscillatory.

In the present study, some of important results for the phase angle have been given. This is important for the calculations of the components of velocity inside the boundary layer. From exponential flow, the results for the oscillatory motions have been deduced and some variations are presented graphically.

BASIC EQUATIONS

The boundary layer equations in two dimensions are

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \tag{1}$$

$$\frac{\partial p}{\partial y} = 0 \tag{2}$$

The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

If $U(x, t)$ is the free stream velocity, we have for the pressure term

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} \tag{4}$$

The boundary conditions are

$$\left.\begin{array}{lll} y = 0: & u = 0, & v = -v_0 \\ y = \infty: & u = U(x, t) & \end{array}\right\} \tag{5}$$

where v_0 is constant non-zero negative suction velocity.

APPLICATIONS TO CIRCULAR CYLINDER

Let the fluid be at rest when the time, $t = 0$ and the cylinder be set into motion with velocity

$$V(t) = A\exp(Ct) \tag{6}$$

The velocity components, u and v are given by

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x} \tag{7}$$

and ψ, the stream function is given by

$$\psi = v_0 x + \sqrt{\frac{\nu}{c}}\exp(Ct)\,U(x)\Big[\chi_0(s) + S\frac{dU}{dx} + \chi_1(s) + S^2\Big\{\Big(\frac{dU}{dx}\Big)^2\chi_{21}(s) + U\frac{d^2U}{dx^2}\chi_{22}(s)\Big\} + \ldots\Big] \tag{8}$$

where

$$s = \sqrt{\frac{C}{\nu}}\,y, \quad S = \int_{-\infty}^{t} V(t)\,dt \tag{9}$$

As usual substituting above equation into (1) and equating the coefficients of various power of S we get

$$\left.\begin{array}{l} \chi_0''' + K\chi_0'' - \chi_0' + 1 = 0 \\ \chi_1''' + K\chi_1'' - 2\chi_1' - \chi_0'^2 + \chi_0\chi_0'' + 1 = 0 \\ \chi_{21}''' + K\chi_{21}'' - 3\chi_{21}' - 2\chi_0'\chi_1' + \chi_0\chi_0'' + \chi_1\chi_0'' = 0 \\ \chi_{22}''' + K\chi_{22}'' - 3\chi_{22}' - \chi_0'\chi_1' + \chi_1\chi_0'' = 0 \end{array}\right\} \tag{10}$$

where

$$K = v_0/\sqrt{(\nu c)} \tag{11}$$

The reduced boundary conditions are

$$\left.\begin{array}{l} s = 0: \chi_0 = \chi_0' = \chi_1 = \chi_1' = \ldots\ldots = 0 \\ s = \infty: \chi_0' = 1, \chi_1' = \chi_{21}' = \ldots\ldots = 0 \end{array}\right\} \tag{12}$$

Above equations are simple and their solutions for the present purpose are

$$\chi_0(s) = s + \frac{1}{h}\Big[\exp(-hs) - 1\Big] \tag{13}$$

$$\chi_1(s) = \frac{1+Kh}{m}\Big[1 - \exp(-ms) - s\exp(-hs) + K\Big\{\exp(-hs) - 1\Big\}\Big] \tag{14}$$

where

$$\left.\begin{aligned} h &= \frac{1}{2}\left(K + \sqrt{K^2+4}\right) \\ \text{and} \quad m &= \frac{1}{2}\left(K^2 + \sqrt{K^2+8}\right) \end{aligned}\right\} \tag{15}$$

DISCUSSIONS

Lal[6] has considered two cases when K is small or large. From these results the cases for small and large frequency of oscillations have also been considered with the help of above equations and replacing C by $i\omega$ where ω is the frequency of the oscillations of the cylinder.

For small values of K, we know[6] that

$$\chi_0 \simeq s^2\left(\frac{1}{2} + \frac{K}{4} + \frac{K^2}{16} - \frac{K^4}{256} + \ldots\right) - s^3\left(\frac{1}{6} + \frac{K}{6} + \frac{K^2}{12} + \frac{K^3}{48} - \frac{K^5}{768} + \ldots\right) \tag{16}$$

$$\chi_0 \simeq s + s^2\left[-0{\cdot}71 + 0{\cdot}25\,K + 0{\cdot}46\,K^2 + 0{\cdot}25\,K^3 + 0{\cdot}0545\,K^4 + 0{\cdot}0063\,K^6 + \ldots\right] \tag{17}$$

and for large values of K,

$$\chi_0 = 0{\cdot}5\,s^2\left(K + \frac{1}{K} - \frac{1}{K^3}\right) \tag{18}$$

$$\chi_1(s) = s^2\left(\frac{2}{K} + \frac{3}{K^3} - \frac{3}{2K^5}\right) \tag{19}$$

OSCILLATORY FLOW

To get an oscillatory motion, C has to be replaced by $i\omega$ in the above equations. From equation (11) we observe that if C is small, then K is large and similarly for large values of C we have small values of K. At the same time, C and ω are increasing or decreasing together. For small and large values of ω, expressions for $\chi_0(s)$ and $\chi_1(s)$ were deduced by Lal[6] and in the present note, these have been utilized for the graphical representations. From such calculations, the phase angles have been calculated which is necessary to find out if the oscillatory motion lags behind or leads the superimposed motion on the cylinder. In Tables 1 and 2, the amplitude of the factor which multiplies the oscillatory component has been calculated. Such calculations are important for determining the velocity components inside the boundary layer.

Case I: *When ω is small i.e. K is large*

In this case, we have (ref. 6; equations 30, 31)

$$\chi_0(s) \simeq 0\cdot 35\, y^2 \Big[\Big\{ v_0^{-3/2} \nu^{\frac{1}{2}} \omega - \nu^{-\frac{1}{2}} v_0^{-1} \omega^{3/2} + \nu^{\frac{1}{2}} v_0^{-3} \omega^{5/2} \Big\} + i \Big\{ v_0^{-3/2} \nu^{\frac{1}{2}} \omega + v_0^{-1} \nu^{-\frac{1}{2}} \omega^{3/2} + v_0^{-3} \nu^{\frac{1}{2}} \omega^{5/2} \Big\} \Big] \quad (20)$$

$$\chi_1(s) \simeq y^2 \Big[- \Big\{ 1\cdot 42\, v_0^{-1} \nu^{-\frac{1}{2}} \omega^{3/2} + 2\cdot 13\, v_0^{-3/2} \nu^{\frac{1}{2}} \omega^{5/2} + 1\cdot 065\, v_0^{-5} \nu^{3/2} \omega^{7/2} \Big\} + i \Big\{ 1\cdot 42\, v_0^{-1} \nu^{-\frac{1}{2}} \omega^{3/2} - 2\cdot 13\, v_0^{-3} \nu^{\frac{1}{2}} \omega^{5/2} + 1\cdot 065\, v_0^{-5} \nu^{3/2} \omega^{7/2} \Big\} \Big] \quad (21)$$

Thus, we may write

$$\chi_0(s) \simeq 0\cdot 35\, y^2 \,|\, A_1 \,|\, \Big\{ \cos\alpha + i \sin\alpha \Big\} \quad (22)$$

where

$$\left.\begin{aligned} |\, A_1 \,| &= \sqrt{A_{1r}^2 + A_{1i}^2} \\ A_{1r} &= v_0^{-3/2} \nu^{\frac{1}{2}} \omega - \nu^{-\frac{1}{2}} v_0^{-1} \omega^{3/2} + \nu^{\frac{1}{2}} v_0^{-3} \omega^{5/2} \\ A_{1i} &= v_0^{-3/2} \nu^{\frac{1}{2}} \omega + \nu^{-\frac{1}{2}} v_0^{-1} \omega^{3/2} + \nu^{\frac{1}{2}} v_0^{-3} \omega^{5/2} \\ \alpha &= \tan^{-1} \frac{A_{1i}}{A_{1r}} \end{aligned}\right\} \quad (23)$$

I have already shown[6], the variation of angle α with ω and thus here other vriations have been shown whose use is required for further development of the boundary layer theory.

Equation (21) may be written as

$$\chi_1(s) \simeq y^2 \,|\, B_1 \,|\, \Big\{ \cos\beta + i \sin\beta \Big\} \quad (24)$$

TABLE 1

VALUES OF $|\,B_1\,|$ FOR INCREASING VALUE OF ω WHEN $\nu = 0\cdot 01$ AND $v_0 = 0\cdot 1$ OR $1\cdot 0$

ω	0·0	0·2	0·4	0·6	0·8	1·0
$\|B_1\|$ when $v_0 = 0\cdot 1$	0	18·5	56·4	117·0	211·5	351·2
$\|B_1\|$ when $v_0 = 1\cdot 0$	0	1·8	5·0	9·4	14·5	19·7

where

$$\left.\begin{aligned} |B_1| &= \sqrt{B_{1r}^2 + B_{1i}^2} \\ B_{1r} &= -\left\{1.42\, v_0^{-1}\, \nu^{-\frac{1}{2}}\, \omega^{3/2} + 2.13\, v_0^{-3/2}\, \nu^{\frac{1}{2}}\, \omega^{5/2} + 1.065\, v_0^{-5}\, \nu^{3/2}\, \omega^{7/2}\right\} \\ B_{1i} &= 1.42\, v_0^{-1}\, \nu^{-\frac{1}{2}}\, \omega^{3/2} - 2.13\, v_0^{-3}\, \nu^{\frac{1}{2}}\, \omega^{5/2} + 1.065\, v_0^{-5}\, \nu^{3/2}\, \omega^{7/2} \\ \beta &= \tan^{-1} \frac{B_{1i}}{B_{1r}} \end{aligned}\right\} \qquad (25)$$

In Table 1, we have calculated the values of $|B_1|$ with the help of B_{1r} and B_{1i} when $\nu = 0.01$ and $v_0 = 0.1$ or 1.0. We find from Table 1 that as ω increases, the magnitude of B_1 also increases. For small values of the suction velocity, the magnitude of B_1 is found to be large.

From Fig. 1, we see that as the suction velocity increases, the phase angle β is found to be increasing for given frequency of oscillations. The phase angle, is found to be decreasing as the frequency of oscillations increases. From the calculations, we have β as negative angle which means that the fluctuations are laging behind the oscillations of the cylinder.

Case II : For large ω i.e. small values of K

In this case from ref. 6, equations 28 and 29, we have

$$\left.\begin{aligned} \chi_0(s) \simeq y^2 & \Big[\Big\{-0.00288\, v_0^4 \nu^{-3} \omega^{-1} + 0.0625\, v_0^4 \nu^{-2} + 0.18\, v_0 \nu^{-3/2} \omega^{\frac{1}{2}}\Big\} \\ & + i \Big\{0.00288\, v_0^4 \nu^{-3} \omega^{-1} + 0.18\, v_0 \nu^{-3/2} \omega^{\frac{1}{2}} + 0.5\, \omega \nu^{-1}\Big\}\Big] \\ & - y^3 \Big[\Big\{-0.11857\, \nu^{-3/2} \omega^{3/2} + 0.5893\, \nu^{-3/2} v_0^2 \omega^{\frac{1}{2}} + 0.021\, v_0^3 \nu^{-3}\Big\} \\ & + i \Big\{0.0013\, v_0^5 \nu^{-4} \omega^{-1} + 0.05893\, \nu^{-5/2} v_0^2 \omega^{\frac{1}{2}} + 0.167\, v_0 \nu^{-2} \omega \\ & + 0.11857\, \nu^{-3/2} \omega^{3/2}\Big\}\Big] \end{aligned}\right\} \qquad (26)$$

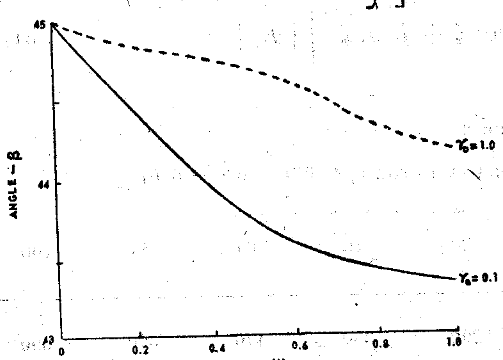

Fig. 1—Variation of angle $-\beta$ with ω for $\nu = 0.01$ and $v = 0.1$ or 1.0.

$$\begin{aligned} \chi_1(s) \simeq y\, e^{i\pi/4} \nu^{-\frac{1}{2}} \omega^{\frac{1}{2}} + y^2 & \Big[\Big\{-0.0063 v_0^6 \nu^{-4} \omega^{-2} + 0.18 v_0^3 \nu^{-5/2} \omega^{-\frac{1}{2}} \\ & + 0.46 v_0^2 \nu^{-2} + 0.18 v_0 \nu^{-3/2} \omega^{\frac{1}{2}}\Big\} + i \\ & \Big\{-0.0545 v_0^4 \nu^{-3} \omega^{-1} - 0.18 v_0^3 \nu^{-5/2} \omega^{-\frac{1}{2}} + \\ & 0.18 v_0 \nu^{-3/2} \omega^{\frac{1}{2}} - 0.71 \nu^{-1} \omega\Big\}\Big] \end{aligned} \qquad (27)$$

From above equations (26) and (27), we easily see that the above two equations may be easily put as

$$\chi_0(s) \simeq y^2 \left[\cos\gamma + i \sin\gamma \right] | C_1 | + y^3 \left[\cos\xi + i \sin\xi \right] | D_1 | \quad (28)$$

where

$$\left.\begin{aligned} | C_1 | &= \sqrt{C_{1r}^2 + C_{1i}^2} \\ C_{1r} &= -0{\cdot}00288\, v_0^4 \nu^{-3} \omega^{-1} + 0{\cdot}0625\, v_0^4 \nu^{-2} + 0{\cdot}18\, v_0 \nu^{-3/2} \omega^{\frac{1}{2}} \\ C_{1i} &= 0{\cdot}00288\, v_0^4 \nu^{-3} \omega^{-1} + 0{\cdot}18\, v_0 \nu^{-3/2} \omega^{\frac{1}{2}} \omega + 0{\cdot}5\, \omega\, \nu^{-1} \\ \gamma &= \tan^{-1} \frac{C_{1i}}{C_{1r}} \end{aligned}\right\} \quad (29)$$

and

$$\left.\begin{aligned} | D_1 | &= \sqrt{D_{1r}^2 + D_{1i}^2} \\ D_{1r} &= 0{\cdot}11857\, \nu^{-3/2} \omega^{3/2} - 0{\cdot}5893\, \nu^{-3/2} v_0^2 \omega^{\frac{1}{2}} - 0{\cdot}021\, v_0^3 \nu^{-3} \\ D_{1i} &= -\Big\{ 0{\cdot}0013\, v_0^5 \nu^{-4} \omega^{-1} + 0{\cdot}05893\, \nu^{-5/2} v_0^2 \omega^{\frac{1}{2}} + 0{\cdot}167\, v_0 \nu^{-2} \omega \\ &\quad + 0{\cdot}11857\, \nu^{-3/2} \omega^{3/2} \Big\} \\ \xi &= \tan^{-1} \frac{D_{1i}}{D_{1r}} \end{aligned}\right\} \quad (30)$$

and

$$\chi_1(s) \simeq y e^{i\pi/4} \nu^{-\frac{1}{2}} \omega^{\frac{1}{2}} + y^2 \left[\cos\zeta + i \sin\zeta \right] | E_1 | \quad (31)$$

TABLE 2

VARIATION IN $| C_1 |$ AND $| E_1 |$ AS ω INCREASES FOR $v_0 = 0{\cdot}1$ AND $\nu = 0{\cdot}01$

ω	16	25	36	49	64	81	100
$\| C_1 \|$	168	227	290	380	470	570	690
$\| E_1 \|$	1070	1690	2455	3360	4410	5605	6925

where

$$\left.\begin{aligned} |E_1| &= \sqrt{E_{1r}^2 + E_{1i}^2} \\ E_{1r} &= -0{\cdot}0063\, v_0^6 \nu^{-4} \omega^{-2} + 0{\cdot}18\, v_0^3 \nu^{-5/2} \omega^{-\frac{1}{2}} + 0{\cdot}46\, v_0^2 \nu^{-2} + 0{\cdot}18\, v_0 \nu^{-3/2} \omega^{\frac{1}{2}} \\ E_{1i} &= -0{\cdot}0545\, v_0^4 \nu^{-3} \omega^{-1} - 0{\cdot}18\, v_0^3 \nu^{-5/2} \omega^{-\frac{1}{2}} + 0{\cdot}18\, v_0 \nu^{-3/2} \omega^{-\frac{1}{2}} - 0{\cdot}71\, \nu^{-1} \omega \\ \zeta &= \tan^{-1} \frac{E_{1i}}{E_{1r}} \end{aligned}\right\} \quad (32)$$

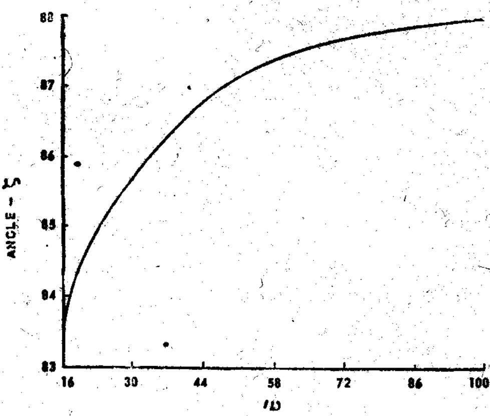

Fig. 2—Variation of $+\gamma$ with ω.

In Table 2, the variation of $|C_1|$ and $|E_1|$ has been shown, for $v_0 = 0{\cdot}1$, and $\nu = 0{\cdot}01$. The values of $|D_1|$ are found to be increasing by increasing the frequency of oscillations. But the values of $|D_1|$ become very large and hence the calculations for $|D_1|$ have been avoided. From Table 2, we see that as ω increases, the amplitudes of C_1 and E_1 increases rapidly. As the numerals are large they have been given in round numbers.

In Fig. 2, the variation of angle $+\gamma$ with ω has been shown. It has been found that the value of γ is positive for the various values of ω. Hence the fluctuation leads by an angle γ. The value of γ increases as the frequency of oscillations increases.

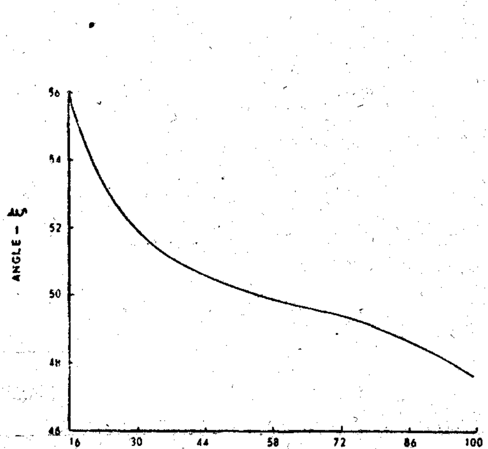

Fig. 3—Variation of angle $-\xi$ with ω when $\nu = 0{\cdot}01$ and $v_0 = 0{\cdot}1$.

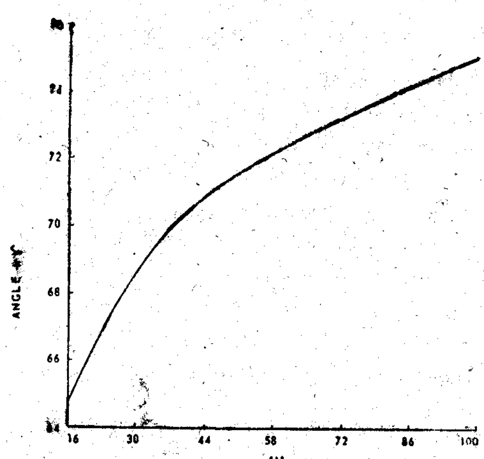

Fig. 4—Variation of $-\zeta$ with ω for $\nu = 0{\cdot}01$ and $v_0 = 0{\cdot}1$.

From Fig. 3, we see that for ω between 25 and 81, the variations are slow. As ω increases beyond 81, angle ξ decreases fast. We easily see that ξ decreases as ω increases.

From Fig. 4, we see that as ω increases, the phase angle also increases. For $\omega > 64$, the increment is slow.

From equations (7) and (8), we have the expression for u inside the boundary layer as

$$u \simeq \exp(Ct)\ U(x) \left[\chi'_0(s) + S \frac{dU}{dx} \chi'_1(s) \right] + \dots \tag{33}$$

which for the oscillatory motion when the frequency of oscillations is small, becomes

$$u \simeq U(x)\ y \left[|M|\, e^{i(\omega t+\eta)} + \frac{dU}{dx} |N|\, e^{i(2\omega t+\delta)} \right] \tag{34}$$

where

$$\left.\begin{aligned} |M| &= \sqrt{M_r^2 + M_i^2} \\ |N| &= \sqrt{N_r^2 + N_i^2} \\ M_r &= v_0 \nu^{-1} + v_0^{-3} \nu \omega^2 \\ M_i &= \omega v_0^{-1} \\ N_r &= 4v_0^{-1} + 3\omega^2 \nu^2 v_0^{-5} \\ N_i &= 6v_0^{-3} \nu \\ \eta = \tan^{-1}\frac{M_i}{M_r}&, \quad \delta = \tan^{-1}\frac{N_i}{N_r} \end{aligned}\right\} \tag{35}$$

and $A = 1$, for convenience of calculations.

We see in this case that the cylinder is moving with velocity $A \cos \omega t$, and the expression for u is obtained by taking the real parts in (34). The above-calculated values of $\chi_0(s)$ and $\chi_1(s)$ may directly be used and put into equation (34) for calculating u inside the boundary layer.

The present note is an extension of my previous paper[6] written under the guidance of late Prof. A. C. Banerji with whom I worked during vacation under a U.G.C. Scheme for university teachers.

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