ON AN INTEGRAL TRANSFORM AND SELF-RECIPROCAL FUNCTIONS

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Bhise's transform generalised by Verma is used to develop the theory of self-reciprocal functions. The Mellin transform is used. Some theorms analogous to Agarwal[1] have been formed.

Recently Verma[2] defined generalised Hankel transform by the relation

$$F(x) = \int_0^\infty G_{2,4}^{2,1}\left(xy \left| \begin{matrix} k-m-\frac{1}{2}-\frac{\nu}{2}, -k+m+\frac{1}{2}+\frac{\nu}{2} \\ \frac{\nu}{2}-\lambda-m, \frac{\nu}{2}-\lambda+m, -\frac{\nu}{2}+\lambda+m, -\frac{\nu}{2}+\lambda-m \end{matrix} \right. \right) f(y)dy \qquad (1)$$

where m is neither an integer nor zero.

Generalised Hankel transform introduced by Bhise[3] is a particular case of (1) when $\lambda = -m$. Thus in turn it includes all the transforms as special cases that are particular cases of the transform given by Bhise, which in turn reduces to Hankel transform in Tricomi's form with $k + m = \frac{1}{2}$.

As this is similar to the $\phi_{\nu, k, \lambda, m}$ —transform of Saxena[4] we call $F(x)$ as $\phi_{\nu\, k, \lambda, m}$ —transform of $f(x)$. When $F(x) = f(x)$ then $f(x)$ shall be called self-reciprocal in $\varphi_{\nu, k, \lambda, m}$ —transform and we shall say that $f(x)$ is $R_\nu(k, \lambda, m)$.

CONDITIONS OF SELF-RECIPROCITY

A function $f(x)$ will be said to be self-reciprocal in the Generalised Hankel transform of order ν if it satisfies the integral equation

$$f(x) = \int_0^\infty G_{2,4}^{2,1}\left(xu \left| \begin{matrix} k-m-\frac{1}{2}-\frac{\nu}{2}, -k+m+\frac{1}{2}+\frac{\nu}{2} \\ \frac{\nu}{2}-\lambda-m, \frac{\nu}{2}-\lambda+m, -\frac{\nu}{2}+\lambda+m, -\frac{\nu}{2}+\lambda-m \end{matrix} \right. \right) \times f(u)\,du \qquad (2)$$

We shall denote such a function by the symbol $R_\nu(k, \lambda, m)$.

If $M(r)$ denotes the Mellin transform of $f(x)$ then we have

$$M(r) = \int_0^\infty f(x)\, x^{r-1}\, dx \qquad (3)$$

Substituting the expression for $f(x)$ in (3) from (2), we obtain*

$$M(r) = \frac{\Gamma\left(\frac{3}{2} - k + m + \frac{\nu}{2} - r\right)\ \Gamma_*\left(\frac{\nu}{2} - \lambda \pm m + r\right)}{\Gamma\left(\frac{\nu}{2} - k + m + \frac{1}{2} + r\right)\ \Gamma_*\left(\frac{\nu}{2} + 1 - r - \lambda \pm m\right)}\ M(1 - r) \qquad (4)$$

it being assumed that (2) is absolutely convergent.
Hence

$$f(x) = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \frac{\Gamma_*\left(\frac{\nu}{2} - \lambda \pm m + r\right)}{\Gamma\left(\frac{1}{2} - k + m + \frac{\nu}{2} + r\right)}\ \psi(r)\, x^{-r}\, dr \qquad (5)$$

where

$$\psi(r) = \frac{M(r)\ \Gamma\left(\frac{1}{2} - k + m + \frac{\nu}{2} + r\right)}{\Gamma_*\left(\frac{\nu}{2} + r - \lambda \pm m\right)} \qquad (6)$$

and satisfies the functional equation

$$\psi(r) = \psi(1 - r) \qquad (7)$$

The existence of the relation (5) demands that $|M(r)| \to 0$ as t tends to infinity, where $r = \sigma + it$. From (6) this implies that

$$\left| \frac{\Gamma_*\left(\frac{\nu}{2} - \lambda \pm m + r\right) \psi(r)}{\Gamma\left(\frac{1}{2} - k + m + \frac{\nu}{2} + r\right)} \right| \to 0$$

as $t \to \infty$

Using asymptotic expression for the Gamma-function, we see that

$$\psi(r) = O\left(e^{\left(\frac{\pi}{2} - \alpha + \eta\right) |t|} \right) \qquad (8)$$

for every positive η however small and $0 < \alpha \leqslant \pi$.

FORMULAE CONNECTING DIFFERENT CLASSES OF SELF-RECIPROCAL FUNCTIONS

Theorem 1 :— If $f(x)$ is $R_\rho\ (h, \mu, n)$ and

$$K(x) = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \frac{\Gamma_*\left(\frac{\rho}{2} - \mu \pm n + r\right)\ \Gamma_*\left(\frac{\nu}{2} - \lambda \pm m + r\right)}{\Gamma\left(\frac{1}{2} - h + n + \frac{\rho}{2} + r\right)\ \Gamma\left(\frac{1}{2} - k + m + \frac{\nu}{2} + r\right)} \qquad (9)$$

$$\times\ \chi(r)\, x^{-r}\, dr\,,$$

where $\qquad \chi(r) = \chi(1 - r),$

then $\qquad g(x) = \int_0^\infty K(xu) f(u)\, du$

is $\quad R_\nu^*(k, \lambda, m)$

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* $\Gamma(a \pm b) = \Gamma(a + b)\, \Gamma(a - b)$.

Proof :—We have

$$g(x) = \int_0^\infty K(xu)\, f(u)\, du$$

Substituting the value of $f(x)$ from (5), we get

$$g(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma_*\left(\frac{\rho}{2} - \mu \pm n + r\right)}{\Gamma\left(\frac{1}{2} - h + n + \frac{\rho}{2} + r\right)}\, \psi(r)\, dr$$

$$\times \int_0^\infty K(xu)\, u^{-r}\, du$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\psi(r)\, x^{r-1}\, \Gamma_*\left(\frac{\rho}{2} - \mu \pm n + r\right)}{\Gamma\left(\frac{1}{2} - h + n + \frac{\rho}{2} + r\right)}\, dr$$

$$\times \int_0^\infty K(l)\, l^{-r}\, dl$$

Now applying Mellin inversion formula to (9) and changing r to $1-r$, and putting

$$\int_0^\infty K(l)\, l^{-r}\, dl = \frac{\Gamma_*\left(\frac{\rho}{2} + 1 - \mu \pm n - r\right)\,\Gamma_*\left(\frac{\nu}{2} + 1 - \lambda \pm m - r\right)}{\Gamma\left(\frac{3}{2} - h + n + \frac{\rho}{2} - r\right)\,\Gamma\left(\frac{3}{2} - k + m + \frac{\nu}{2} - r\right)}\,\chi(r)$$

we get

$$g(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{\Gamma_*\left(\frac{\nu}{2} - \lambda \pm m + r\right)}{\Gamma\left(\frac{1}{2} - k + m + \frac{\nu}{2} + r\right)}\, \phi(r)\, x^{-r}\, dr,$$

since

$$\phi(r) = \frac{\Gamma_*\left(\frac{\rho}{2} - \mu \pm n + r\right)\,\Gamma_*\left(\frac{\rho}{2} + 1 - \mu \pm n - r\right)}{\Gamma\left(\frac{1}{2} - h + n + \frac{\rho}{2} + r\right)\,\Gamma\left(\frac{3}{2} - h + n + \frac{\rho}{2} - r\right)} \times$$

$$\times\, \psi(1-r)\, \chi(1-r)$$

$$= \phi(1-r)$$

We see that $g(x)$ is $R_\nu(k, \lambda, m)$, which proves the theorem.

Corollary 1.

If in the above theorem we put $h = k$, $\mu = \lambda$, $n = m$, then

$$K(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma_*\left(\frac{\rho}{2} - \lambda \pm m + r\right)\,\Gamma_*\left(\frac{\nu}{2} - \lambda \pm m + r\right)}{\Gamma\left(\frac{1}{2} - k + m + \frac{\rho}{2} + r\right)\,\Gamma\left(\frac{1}{2} - k + m + \frac{\nu}{2} + r\right)}$$

$$\times\, \chi_1(r)\, x^{-r}\, dr,$$

where $\chi_1(r) = \chi_1(1-r)$,

is a kernel transforming $R_\rho(k, \lambda, m)$ into $R_\nu(k, \lambda, m)$

Theorem 2. If $f(x)$ is $R_\rho(h, \mu, n)$ and

$$K(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma_*\left(\frac{\rho}{2} - \mu \pm n + r\right)\ \Gamma_*\left(\frac{\nu}{2} + 1 - r - \lambda \pm m\right)}{\Gamma\left(\frac{3}{2} - h + n + \frac{\rho}{2} + r\right)\ \Gamma\left(\frac{3}{2} - k + m + \frac{\nu}{2} - r\right)} \times \chi(r)\, x^{-r}\, dr \tag{10}$$

where

$$\chi(r) = \chi(1-r)$$

then

$$g(x) = \int_0^\infty f(xu)\, K(u)\, du \text{ is } R_\nu(k, \lambda, m)$$

The proof of this theorem follows exactly in a manner similar to that of Theorem 1.

Example. To illustrate the theorem we put

$$\chi(r) = \Gamma\left(\frac{\nu}{2} + r\right)\ \Gamma\left(1 + \frac{\nu}{2} - r\right) \text{ in (10)}$$

Then

$$K(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma_*\left(\frac{\rho}{2} - \mu \pm n + r\right)\ \Gamma_*\left(\frac{\nu}{2} + 1 - r - \lambda \pm m\right)}{\Gamma\left(\frac{1}{2} - h + n + \frac{\rho}{2} + r\right)\ \Gamma\left(\frac{3}{2} - k + m + \frac{\nu}{2} - r\right)} \times \Gamma\left(\frac{\nu}{2} + r\right)\ \Gamma\left(1 + \frac{\nu}{2} - r\right) x^{-r}\, dr$$

which on evaluation gives

$$G^{3,3}_{4,4}\left(x \,\middle|\, \begin{matrix} \lambda - m - \frac{\nu}{2}, \lambda + m - \frac{\nu}{2}, -\frac{\nu}{2}, \frac{1}{2} - h + n + \frac{\rho}{2} \\ \frac{\rho}{2} - \mu + n, \frac{\rho}{2} - \mu - n, \frac{\nu}{2}, -\frac{1}{2} + k - m - \frac{\nu}{2} \end{matrix}\right)$$

a kernel transforming $R_\rho(h, \mu, n)$ into $R_\nu(k, \lambda, m)$ provided that $R(\rho - 2\mu + 2n) > 0$, $R(\rho - 2\mu - 2n) > 0$, $R(\nu) > 0$; $2m$, $2n$ neither integer nor zero.

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