SOME CONTOUR INTEGRALS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTION

S. D. Bajpai

Shri G. S. Technological Institute, Indore*

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Contour integral involving Fox's H-Function and modified Bessel function of the first kind has been calculated. Some important properties and particular cases of H-function, which is a generalization of G-function, have been derived and discussed.

In this paper we have evaluated a contour integral involving Fox's H-function and modified Bessel function of the first kind. On specializing the parameters the integral yields many results scattered throughout the literature, some of which are given by MacRobert[1] and Meijer[2].

The H-function introduced by Fox[3] will be represented and defined as follows:

$$H_{p,\,q}^{m,\,n}\left[z \left| \begin{matrix} (a_1, e_1), \ldots, (a_p, e_p) \\ (b_1, f_1), \ldots, (b_q, f_q) \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - f_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + e_j s)\, z^s}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + f_j s) \prod_{j=n+1}^{p} \Gamma(a_j - e_j s)} ds, \tag{1}$$

where an empty product is interpreted as $1, 0 \leqslant m \leqslant q, 0 \leqslant n \leqslant p$; e's and f's are all positive; L is a suitable contour of Barnes type such that the poles of $\Gamma(b_j - f_j s), j = 1, \ldots\ldots, m$ lie on the right hand side of the contour and those of $\Gamma(1 - a_j + e_j s), j = 1, \ldots\ldots, n$ lie on the left hand side of the contour.

Recently Braaksma[4] has discussed, asymptotic expansions and analytic continuations for the H-function.

Now we discuss some important properties and particular cases of the H-function, which are apparent from the definition of the H-function.

The H-function is symmetric in pairs $(a_1, e_1), \ldots\ldots\ldots\ldots, (a_n, e_n)$ likewise in $(a_{n+1}, e_{n+1}), \ldots\ldots\ldots\ldots, (a_p, e_p)$; in $(b_1, f_1), \ldots\ldots\ldots, (b_m, f_m)$ and in $(b_{m+1}, f_{m+1}), \ldots\ldots\ldots, (b_q, f_q)$.

In one of (a_j, e_j) $(j = 1, \ldots\ldots, n)$ is same as one of (b_j, f_j) $(j = m+1, \ldots\ldots, q)$ or one of (b_j, f_j) $(j = 1, \ldots\ldots, m)$ is same as one of (a_j, e_j) $(j = n+1, \ldots\ldots, p)$ then the H-function reduces to one of lower order, i.e., each of p, q and n or m decreases by unity.

* Present address :—Regional Engineering College, Kurukshetra.

In what follows for sake of brevity

$$\sum_{j=1}^{p} e_j - \sum_{j=1}^{q} f_j \equiv A ,$$

$$\sum_{j=1}^{n} e_j - \sum_{j=n+1}^{p} e_j + \sum_{j=1}^{m} f_j - \sum_{j=m+1}^{q} f_j \equiv B$$

and (a_p , e_p) represents the set of parameters $(a_1 , e_1) , \ldots\ldots , (a_p , e_p)$.

$$z^{\sigma} H^{m,n}_{p,q}\left[z \left| \begin{matrix} (a_p , e_p) \\ (b_q , f_q) \end{matrix} \right. \right] = H^{m,n}_{p,q}\left[z \left| \begin{matrix} (a_p + \sigma e_p , e_p) \\ (b_q + \sigma f_q , f_q) \end{matrix} \right. \right] \tag{2}$$

$$H^{m,n}_{p,q}\left[z \left| \begin{matrix} (a_p , e_p) \\ (b_q , f_q) \end{matrix} \right. \right] = H^{n,m}_{q,p}\left[z^{-1} \left| \begin{matrix} (1 - b_q , f_q) \\ (1 - a_p , e_p) \end{matrix} \right. \right] \tag{3}$$

$$H^{m,n}_{p,q}\left[z \left| \begin{matrix} (a_p , 1) \\ (b_q , 1) \end{matrix} \right. \right] = G^{m,n}_{p,q}\left[z \left| \begin{matrix} a_p \\ b_q \end{matrix} \right. \right], \tag{4}$$

where the right hand side is Meijer's G-function.

$$H^{p,1}_{q+1,p}\left[z \left| \begin{matrix} (1 , 1) , (b_q , 1) \\ (a_p , 1) \end{matrix} \right. \right] = E\left[\begin{matrix} a_p : z \\ b_q \end{matrix} \right], \tag{5}$$

where the right hand side is MacRobert's E-function.

$$H^{1,p}_{p,q+1}\left[x \left| \begin{matrix} (1 - a_p , e_p) \\ (0 , 1) , (1 - b_q , f_q) \end{matrix} \right. \right] = \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + e_j r)}{\prod_{j=1}^{q} \Gamma(b_j + f_j r)} \cdot \frac{(-x)^r}{r!} \tag{6}$$

The above series was studied by Wright[5] and has been called as Wright's generalized hypergeometric function and is denoted by the symbol

$${}_p\psi_q\left[\begin{matrix} (a_p , e_p) \\ (b_q , f_q) \end{matrix} ; -x \right]$$

$$H^{2,0}_{1,2}\left[z \left| \begin{matrix} (\lambda - k + 1 , 1) \\ (1/2 + \lambda + \mu , 1) , (1/2 + \lambda - \mu , 1) \end{matrix} \right. \right] = 2^{\lambda} e^{z/2} W_{k,\mu}(z), \tag{7}$$

where $W_{k,\mu}(z)$ is a Whittaker function.

$$H^{1,0}_{0,2}\left[z \left| (0 , 1) , (-\nu , \mu) \right. \right] = \sum_{r=0}^{\infty} \frac{(-z)^r}{r!\, \Gamma(1 + \nu + \mu r)} = J^{\mu}_{\nu}(z), \tag{8}$$

where $J^{\mu}_{\nu}(z)$ is Maitland's generalized Bessel function[6].

$$H^{2,0}_{0,2}\left[z^2/4 \left| (\lambda/2 - \nu/2 , 1) , (\lambda/2 + \nu/2 , 1) \right. \right] = 2^{1-\lambda} z^{\lambda} K_{\nu}(z), \tag{9}$$

where $K_{\nu}(z)$ is a modified Bessel function.

The following formula is required in the proof:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{1/2-\rho} I_\nu(xy)\,dy = \frac{(x/2)^{\rho-3/2}}{\Gamma\left(\frac{\rho+\nu}{2}+\frac{1}{4}\right)\Gamma\left(\frac{\rho-\nu}{2}+\frac{1}{4}\right)},\ \mathrm{Re}(\rho) > |\mathrm{Re}\,\nu| - 1/2, \quad (10)$$

which follows from reference 7.

The Integral

The integral to be established is

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{1/2-\rho} I_\nu(xy)\, H^{m,n}_{p,q}\left[zy^{2h}\,\middle|\,\begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right] dy$$

$$= (x/2)^{\rho-3/2} H^{m,n}_{p+2,q}\left[z(x/2)^{-2h}\,\middle|\,\begin{matrix}(a_p, e_p), \{(\rho+\nu)/2+1/4, h\},\\ \{(\rho-\nu)/2+1/4, h\}\\(b_q, f_q)\end{matrix}\right] \quad (11)$$

where h is a positive number,

$$A \leqslant 0,\ B > 0,\ |\arg z| < \tfrac{1}{2}B\pi,$$

$$\mathrm{Re}\ [\rho + 2h(1-a_j)/e_j] > |\mathrm{Re}\,\nu| - \tfrac{1}{2} \quad (j = 1, \ldots\ldots, n)$$

Proof;

Expressing the H-function in the integrand as a Mellin-Barnes type integral (1) and interchanging the order of integrations, which is justified due to the absolute convergence of the integrals involved in the process, we have

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - f_j s)\prod_{j=1}^{n}\Gamma(1-a_j+e_j s)\, z^s}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_j s)\prod_{j=n+1}^{p}\Gamma(a_j - e_j s)} \cdot \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{1/2-\rho+2hs} I_\nu(xy)\,dy\,ds.$$

Evaluating the inner-integral with the help of (10), we get

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - f_j s)\prod_{j=1}^{n}\Gamma(1-a_j+e_j s)\, z^s}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_j s)\prod_{j=n+1}^{p}\Gamma(a_j - e_j s)} \cdot \frac{(x/2)^{\rho-2hs-3/2}}{\Gamma\left(\frac{\rho+\nu}{2}+\frac{1}{4}-hs\right)\Gamma\left(\frac{\rho-\nu}{2}+\frac{1}{4}-hs\right)}\,ds.$$

On applying (1), the formula (11) is obtained.

In (11), putting $\nu = -\frac{1}{2}$, $I_{-\frac{1}{2}}(xy) = \left(\frac{2}{\pi xy}\right)^{\frac{1}{2}} e^{xy}$, it reduces to the form

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-\rho} e^{xy} H^{m,n}_{p,q}\left[zy^{2h}\,\middle|\,\begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right] dy$$

$$= \sqrt{\pi}\,(x/2)^{\rho-1} H^{m,n}_{p+2,q}\left[z(x/2)^{-2h}\,\middle|\,\begin{matrix}(a_p, e_p), (\rho/2, h), \{(\rho+1)/2, h\}\\(b_q, f_q)\end{matrix}\right] \quad (12)$$

where h is a positive number,

$$A \leqslant 0, \quad B > 0, \quad |\arg z| < \tfrac{1}{2} B\pi,$$

$$\mathrm{Re}\ [\rho + 2h(1 - a_j)/e_j] > 0 \ (j = 1, \ldots\ldots, n).$$

PARTICULAR CASES

The H-function is a generalization of the G-function, which itself is a generalization of many higher transcendental functions[8]. Therefore, the formulae (11) and (12) are of general character and may encompass several cases of interest. It is interesting to note that the results (11) and (12) lead to generalization of many results of inverse K-transforms[7] and inverse Laplace transforms[9]. However, some interesting particular cases are given below:

(*i*) In (11), substituting $e_j = f_i = 1$ ($j = 1, \ldots\ldots, n; i = 1, \ldots, m$), assuming h as a positive integer, using (4) and simplifying with the help[8] of (1), we obtain

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{1/2-\rho} I_\nu(xy)\, G^{m,n}_{p,q}\left[zy^{2h} \middle| \begin{matrix} a_p \\ b_q \end{matrix} \right] dy$$

$$= (2\pi)^{h-1} h^{1/2-\rho} (x/z)^{\rho-3/2} G^{m,n}_{p+2h,q}\left[z\left(\frac{2h}{x}\right)^{2h} \middle| \begin{matrix} a_p, \triangle\{h, (\rho+\nu)/2+1/4\} \\ \triangle\{h, (\rho-\nu)/2+1/4\} \\ b_q \end{matrix} \right] \tag{13}$$

where h is a positive integer, $2(m+n) > p+q$, $|\arg z| < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi$, $\mathrm{Re}\,[\rho + 2h(1-a_j)] > |\mathrm{Re}\,\nu| - \tfrac{1}{2}$ ($j = 1, \ldots\ldots, n$) and the symbol $\triangle(h, \alpha)$ represents the set of parameters

$$\frac{\alpha}{h}, \frac{\alpha+1}{h}, \ldots\ldots, \frac{\alpha+h-1}{h}.$$

In (13), taking $h = 1$, $\rho = 0$ and adjusting the parameters suitably, it reduces to a known result[7].

(*ii*) In (12), replacing $2h$ by h, reducing the H-function to the G-function as above, using duplication formula for the gamma-function[8] and simplifying, we get

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-\rho} e^{xy} G^{m,n}_{p,q}\left[zy^h \middle| \begin{matrix} a_p \\ b_q \end{matrix} \right] dy$$

$$= (2\pi)^{h/2-1/2} h^{1/2-\rho} x^{\rho-1} G^{m,n}_{p+h,q}\left[z\left(\frac{h}{x}\right)^h \middle| \begin{matrix} a_p, \triangle(h, \rho) \\ b_q \end{matrix} \right], \tag{14}$$

where h is a positive integer,

$$2(m+n) > p+q, \ |\arg z| < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,$$

$$\mathrm{Re}\,[\rho + h(1-a_j)] > 0 \ (j = 1, \ldots\ldots, n).$$

In (14), setting $h = 1$, $x = -1$, $\rho = 1-\nu$, $m = 1$, $b_1 = 0$, $n = p$, replacing y by $-y$, q by $q+1$, b_k by b_{k-1} ($k = 2, 3, \ldots\ldots, q+1$), and using reference 10, viz.,

$$G^{1,n}_{p,q}\left[z\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right]=\frac{\prod\limits_{j=1}^{n}\Gamma(1+b_1-a_j)\,z^{b_1}}{\prod\limits_{j=1}^{m}\Gamma(a_j-b_1)}\;{}_p\phi_{q-1}\left[\begin{matrix}1+b_1-a_1,\ldots,1+b_1-a_p;(-1)^{p-q-1}z\\ 1+b_1-b_2,\ldots,1+b_1-b_q\end{matrix}\right],$$

and deforming the contour suitably, it reduces to a result given by Meijer[2].

(*iii*) In (11), adjusting the parameters suitably in view of (5) and simplifying, we get

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{1/2-\rho}\, I_\nu(xy)\, E\left[\begin{matrix}a_p : zy^{2h}\\ b_q\end{matrix}\right]dy$$

$$=(2\pi)^{h-1}\,h^{1/2-\rho}\,(x/2)^{\rho-3/2}\,E\left[\begin{matrix}a_p : z\,(2h/x)^{2h}\\ b_q, \triangle\{h, (\rho+\nu)/2+1/4\}, \triangle\{h, (\rho-\nu)/2+1/4\}\end{matrix}\right] \tag{15}$$

where h is a positive integer, $p+1>q$, $|\arg z|<(p-q+1)\,\pi/2$, $\mathrm{Re}\,\rho>|\mathrm{Re}\,\nu|-\frac{1}{2}$

In (15), putting $h=1$ and adjusting parameters, we get a known result[7].

(*iv*) In (12), replacing $2h$ by h, reducing the H-function to the E-function and simplifying, we have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-\rho}\, e^{xy}\, E\left[\begin{matrix}a_p : zy^{h}\\ b_q\end{matrix}\right]dy$$

$$=(2\pi)^{h/2-1/2}\,h^{1/2-\rho}\,x^{\rho-1}\,E\left[\begin{matrix}a_p : z\,(h/x)^{h}\\ b_q, \triangle(h,\rho)\end{matrix}\right], \tag{16}$$

where h is a positive integer, $p+1>q$, $|\arg z|<(p-q+1)\,\pi/2$, $\mathrm{Re}\;\rho>0$.

In (16), putting $x=1$ and deforming the contour suitably, we obtain a result given by MacRobert[1].

(*v*) In (11), replacing m by 1, n by p, q by $q+1$, a_j by $1-a_j$ $(j=1,\ldots,p)$, b_1 by 0, f_1 by 1, b_j by $1-b_{j-1}$, f_j by f_{j-1} $(j=2,3,\ldots,q)$ and using (6), we get

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{1/2-\rho}\, I\,(xy)\;{}_p\psi_q\left[\begin{matrix}(a_p, e_p);-zy^{2h}\\ (b_q, f_q)\end{matrix}\right]dy$$

$$=(x/2)^{\rho-3/2}\,H^{1,p}_{p+2,q+1}\left[z\left(\frac{x}{2}\right)^{-2h}\,\middle|\,\begin{matrix}(1-a_p, e_p), \{(\rho+\nu)/2+1/4, h\}\\ \{(\rho-\nu)/2+1/4, h\}\\ (0,1), (1-b_q, f_q)\end{matrix}\right] \tag{17}$$

where the conditions are corresponding to (11).

(*vi*) In (11), taking $m=q=2$, $n=0$, $p=1$, $a_1=\lambda-k+1$, $b_1=\frac{1}{2}+\lambda+\mu$, $b_2=\lambda-\mu+\frac{1}{2}$, and $e_1=f_1=f_2=1$ and using (7), we get

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{1/2-\rho} I_\nu(xy)\, e^{zy^{2h}/2}\, W_{k,\mu}(zy^{2h})\,dy$$

$$= 2^{-\lambda}(x/2)^{\rho-3/2} H_{3,2}^{2,0}\left[z(x/2)^{-2h} \middle| \begin{matrix} (\lambda-k+1,1),\{(\rho+\nu)/2+1/4,h\} \\ \{(\rho-\nu)/2+1/4,h\} \\ (\frac{1}{2}+\lambda+\mu,1),(\lambda-\mu+\frac{1}{2},1) \end{matrix} \right] \quad (18)$$

where the conditions are corresponding to (11).

(*vii*) In (11), setting $m=1, n=p=0, q=2, b_1=0, b_2=-u, f_1=1, f_2=\mu$ and using (8), we obtain

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{1/2-\rho} I_\nu(xy)\, J_u^\mu\left(zy^{2h}\right)dy$$

$$=(x/2)^{\rho-3/2} H_{2,2}^{1,0}\left[z(x/2)^{-2h} \middle| \begin{matrix} \{(\rho+\nu)/2+1/4,h\},\{(\rho-\nu)/2+1/4,h\} \\ (0,1),(-u,\mu) \end{matrix} \right], \quad (19)$$

where the conditions are corresponding to (11).

(*viii*) Putting $m=q=2, n=p=0, b_1=\frac{1}{2}\lambda-\frac{1}{2}u, b_2=\frac{1}{2}\lambda+\frac{1}{2}u$ and $f_1=f_2=1$, the integral (11), in view of (9), reduces to

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{1/2-\rho} I_\nu(xy)\, K_u(2\sqrt{z}\,y^h)\,dy$$

$$=2^{-1}(\sqrt{z}y^h)^{-\lambda}(x/2)^{\rho-3/2} H_{2,2}^{2,0}\left[z(x/2)^{-2h} \middle| \begin{matrix} \{(\rho+\nu)/2+1/4,h\} \\ \{(\rho-\nu)/2+1/4,h\} \\ (\lambda/2-u/2,1)\ (\lambda/2+u/2,1) \end{matrix} \right] \quad (20)$$

where the conditions are corresponding to (11).

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REFERENCES

1. MacRobert, T. M., *Proc. Glasgow Math. Assoc.*, 1 (1953), 190.
2. Meijer, C. S., *Nedrel. Akad. Wetensch Proc.*, Ser. A, 56 (1953), 349.
3. Fox, C., *Trans. Amer. Math. Soc.*, **98** (1961), 395.
4. Braaksma, B. L. J., *Compos. Math.*, 15 (1963), 239.
5. Wright, E. M., *J. London Math. Soc.*, 38 (1953), 286.
6. Wright, E. M., *Proc. London Math. Soc.*, 10 (1935), 257.
7. Erdelyi, A., Tables of Integral Transforms, Vol. 2 (McGraw-Hill, New York), 1954.
8. Erdelyi, A., Higher Transcendental Functions, Vol. 1 (McGraw-Hill, New York), 1953.
9. Erdelyi, A., Tables of Integral Transforms, Vol. 1 (McGraw-Hill, New York), 1954.
10. Meijer, C. S., *Nedrel. Akad. Wetensch. Proc.* Ser. A, 55 (1952), 369.