TEMPERATURE DISTRIBUTION IN A HEAT GENERATING SOLID WITH PARALLEL FLUID FLOW

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The temperature distribution in a semi-infinite solid generating heat according to an exponential law, has been obtained by the use of Laplace transformation. The solid is in contact with a viscous incompressible fluid which starts moving parallel to it impulsively with a uniform velocity. In the solution is obtained a non-dimensional number A $(= \mu\, u_0^2/Q_0\, L^2)$. The effect of A on the surface temperature of the solid has been presented graphically.

Johnson[1] has studied a problem of heat transfer from a moving fluid to a solid with the assumptions that the fluid is initially at a constant temperature and is set in motion impulsively with a uniform velocity along the surface of a conducting semi-infinite solid of zero initial temperature. The fluid region is also taken to be semi-infinite. At the solid fluid interface the conditions of continuity of flux and temperature are assumed.

In many practical situations we come across problems of heat generating solid in contact with parallel fluid flow. Such problems, for example, are very common in chemical engineering processes, heat exchange mechanisms and reactor technology. In order to study the effect of flow on the temperatures of the fluid and the solid it is necessary to solve both the flow and heat transfer equations for the fluid together with the conduction equation for the solid.

In this paper we find a solution to the problem of a semi-infinite solid, generating heat according to an exponential law of heat generation, in contact with parallel flow of an incompressible viscous fluid. (Exponential source of heat has already been considered by various authors. Carslaw and Jaeger[2] have considered this type of source in studying some problems of heat generation in semi-infinite solids. Cook[3] and Jeffreys[4] used an exponential source as an approximation to the heating of the body by microwaves and radioactive heat generation respectively). The system is assumed to be at zero temperature initially. The fluid is set in motion impulsively with a uniform velocity along the solid surface. The velocity distribution in this case satisfies the diffusion equation for which the solution is known. This known velocity function is substituted in the heat transfer equation for the fluid and we solve the heat transfer equation for the fluid and the conduction equation for the solid as a conjugate problem assuming the conditions of continuity of flux and temperature at the solid fluid interface. The solution is obtained by Laplace transformation. In the solution of this problem we get a non-dimensional number $A = \left[\dfrac{\mu u_0^2}{Q_0 L^2}\right]$ which is nothing but the ratio of the square of the initial fluid velocity u_0 to the maximum volumetric heat generation Q_0 in the solid, multiplied by a constant factor.

Numerical computation is carried out for the solid region and the results are presented in Fig. 1 and Fig. 2 for various values of the non-dimensional numbers $X\left(=\frac{x}{L}\right)$, $F_0\left(=\frac{k_2 t}{L^2}\right)$ and $A\left(=\frac{\mu u_0^2}{Q_0 L^2}\right)$.

STATEMENT AND MATHEMATICAL FORMULATION OF THE PROBLEM

Consider a semi-infinite heat generating solid $(x < 0)$ initially at zero temperature. The heat generation is at a rate $Q_0 e^{mx}$ where m and Q_0 are constants. The region $x > 0$ is occupied by an incompressible viscous fluid of zero initial temperature and it suddenly starts moving with a uniform velocity u_0 parallel to $x = 0$. The conditions of continuity of flux and temperature are assumed at the solid fluid interface. Neglecting the convective terms in the equations for the fluid, find the analytical solutions for the temperature distributions in the fluid and the solid.

Defining the velocity and temperature of the fluid and temperature of the solid by $u\ (x, t)$, $T_1\ (x, t)$ and $T_2\ (x, t)$ respectively, the above problem can be expressed mathematically by the following partial differential equations:

$$\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} = 0, \quad x > 0, \quad t > 0 \tag{1}$$

$$\rho_1 c_1 \frac{\partial T_1}{t} - K_1 \frac{\partial^2 T_1}{\partial x^2} = \mu \left(\frac{\partial u}{\partial x}\right)^2, \quad x > 0, \quad t > 0 \tag{2}$$

$$\rho_2 c_2 \frac{\partial T_2}{\partial t} - K_2 \frac{\partial^2 T_2}{\partial x^2} = Q_0 e^{mx}, \quad x < 0, t > 0, \quad m \geqslant 0 \tag{3}$$

It is also governed by the following initial conditions and boundary conditions :

Initial Conditions :

$$u = u_0 \tag{4}$$

$$T_1 = T_2 = 0 \tag{5}$$

Boundary Conditions :

$$\left.\begin{aligned} u &= 0 & (6)\\ T_1 &= T_2 & (7)\\ K_1 \frac{\partial T_1}{\partial x} &= K_2 \frac{\partial T_2}{\partial x} & (8)\end{aligned}\right\} \quad x = 0$$

$$\left.\begin{aligned} u &= u_0 & (9)\\ T_1 &= 0 & (10)\end{aligned}\right\} \quad x \to \infty$$

$$T_2 = 0 \qquad x \to -\infty \tag{11}$$

SOLUTION OF THE PROBLEM

The solution of equation (1) subject to the initial condition (4) and boundary conditions (6) and (9) is

$$u = u_0 \operatorname{erf} [x/2\nu t] \tag{12}$$

Substituting this value of u in equation (2), we get

$$\frac{\partial^2 T_1}{\partial x^2} - \frac{1}{k_1}\frac{\partial T_1}{\partial t} = -\frac{a\, u_0^2}{\pi\nu} e^{-x^2/2\nu t} \tag{13}$$

Let $\overline{T}$ denote the Laplace transform of a function $T(x, t)$ so that

$$\overline{T}(x, p) = \int_0^\infty T(x, t)\, e^{-pt}\, dt \tag{14}$$

Multiplying equations (3), (13) and boundary conditions (7) to (11) by e^{-pt} and integrating with respect to t from 0 to ∞ using the initial conditions, we get

$$\frac{d^2 \overline{T}_1}{dx^2} - q_1^2\, \overline{T}_1 = -\frac{2a\, u_0^2}{\pi\nu} K_0\left(\sqrt{\frac{2}{P_r}}\, q_1 x\right) \tag{15}$$

$$\frac{d^2 \overline{T}_2}{dx^2} - q_2^2\, \overline{T}_2 = -\frac{Q_0\, e^{mx}}{K_2 p} \tag{16}$$

$$\left.\begin{aligned} \overline{T}_1 &= \overline{T}_2 & (17)\\ K_1 \frac{d\overline{T}_1}{dx} &= K_2 \frac{d\overline{T}_2}{dx} & (18) \end{aligned}\right\}\quad x = 0$$

$$\overline{T}_1 = 0, \quad x \to \infty \tag{19}$$

$$\overline{T}_2 = 0, \quad x \to -\infty \tag{20}$$

where $P_r = \frac{\nu}{k_1}$, $a = \frac{\mu}{K_1}$, $q_1^2 = \frac{p}{k_1}$, $q_2^2 = \frac{p}{k_2}$

and K_0 is the modified Bessel function of second kind and of zero order.

The solutions of (15) and (16) can be written as

$$\overline{T}_1 = A_1 e^{-q_1 x} - \frac{a u_0^2}{\pi\nu q_1} e^{q_1 x} \int_\infty^x K_0\left(\sqrt{\frac{2}{P_r}}\, q_1 \xi\right) e^{-q_1\xi}\, d\xi$$

$$+ \frac{a u_0^2}{\pi\nu q_1} e^{-q_1 x} \int_0^x K_0\left(\sqrt{\frac{2}{P_r}}\, q_1 \xi\right) e^{q_1\xi}\, d\xi, \quad x > 0 \tag{21}$$

$$\bar{T}_2 = A_2 e^{q_2 x} - \frac{Q_0 e^{mx}}{K_2 p [m^2 - q_2^2]}, \quad x < 0 \tag{22}$$

Solving for A_1 and A_2 using the boundary conditions (17) and (18), we get

$$\bar{T}_1 = \left[\frac{K_1 - K_2\sigma}{K_1 + K_2\sigma} \right] F e^{-q_1 x} + \frac{Q_0 e^{-q_1 x}}{\sigma \left[K_1 + K_2 \sigma \right] p q_1 \left[q_1 + \frac{m}{\sigma} \right]}$$

$$- \frac{a u_0^2 e^{q_1 x}}{\pi \nu q_1} \int_{\infty}^{x} K_0 \left(\sqrt{\frac{2}{P_r}} q_1 \xi \right) e^{-q_1 \xi} d\xi$$

$$+ \frac{a u_0^2}{\pi \nu q_1} e^{-q_1 x} \int_{0}^{x} K_0 \left(\sqrt{\frac{2}{P_r}} q_1 \xi \right) e^{q_1 \xi} d\xi \quad , \quad x > 0 \tag{23}$$

$$\bar{T}_2 = \frac{2 K_1 F e^{q_2 x}}{K_1 + K_2\sigma} + \frac{Q_0 \sigma e^{q_2 x}}{m [K_1 + K_2 \sigma] p q_2}$$

$$+ \frac{Q_0 [K_1 - \sigma K_2] e^{q_2 x}}{2 m K_2 [K_1 + \sigma K_2] p [q_2 + m]} - \frac{Q_0 e^{q_2 x}}{2 m K_2 p [q_2 - m]}$$

$$- \frac{Q_0 e^{mx}}{K_2 p [m^2 - q_2^2]}, \qquad x < 0 \tag{24}$$

where

$$F = \frac{a u_0^2}{\pi \nu q_1} \int_0^{\infty} K_0 \left(\sqrt{\frac{2}{P_r}} q_1 \xi \right) e^{-q_1 \xi} d\xi$$

The inversion of all terms in (23) and (24) is available (see appendix[5] V) except for the last two terms in (23) for which we use the convolution theorem[6] and the last term in (24) which has simple poles at $p=0$ and $p=k_2 m^2$. After taking inversion of all terms in (23) and (24) we can write down the solutions $T_1 (x, t)$ and $T_2 (x, t)$ in the non-dimensional form as follows :

$$\Theta_1 = \left[\frac{AM}{\pi \bar{K}} \left\{ \frac{\bar{K} - \sigma}{\bar{K} + \sigma} \right\} - \frac{\alpha X + \sigma}{[\bar{K} + \sigma] \alpha^2} \right] \operatorname{erfc} \left[X / 2\sigma F_0 \right]$$

$$+ \frac{2\sigma}{\alpha [\bar{K} + \sigma]} \sqrt{\frac{F_0}{\pi}} e^{-X^2/4\sigma^2 F_0} + \frac{\sigma e^{\frac{\alpha}{\sigma} X + F_0 \alpha^2}}{[\bar{K} + \sigma] \alpha^2} \operatorname{erfc} \left[(X/2\sigma F_0) + \alpha \sqrt{F_0} \right]$$

$$+ \frac{A P_r^{1/2}}{2 \pi \bar{K}} \int_0^{F_0} \frac{e^{(-X^2/2\sigma^2) [P_r (F_0 - \xi) + 2\xi]}}{\sqrt{(F_0 - \xi) [P_r (F_0 - \xi) + 2\xi]}} \operatorname{erfc} \left[\frac{- \sqrt{P_r (F_0 - \xi)}\, X}{2\sigma \sqrt{P_r (F_0 - \xi) + 2\xi}} \right] d\xi$$

$$X > 0 \tag{25}$$

$$\Theta_2 = \left[\frac{2AM}{\pi[\bar{K}+\sigma]} + \frac{\bar{K}+\alpha\sigma X}{[\bar{K}+\sigma]}\right] \text{erfc}\,[-X/2\sqrt{F_0}]$$

$$+ \frac{2\sigma}{\alpha[\bar{K}+\sigma]}\sqrt{\frac{F_0}{\pi}}\; e^{-X^2/4F_0}$$

$$- \frac{[\bar{K}-\sigma]}{2\alpha^2[\bar{K}+\sigma]}\; e^{-\alpha X + F_0\alpha^2}\; \text{erfc}\left[-\frac{X}{2\sqrt{F_0}} + \alpha\sqrt{F_0}\right]$$

$$- \frac{e^{\alpha X + F_0\alpha^2}}{2\alpha^2}\text{erfc}\left[-\frac{X}{2\sqrt{F_0}} - \alpha\sqrt{F_0}\right]$$

$$- \left[1 - e^{F_0\alpha^2}\right]\frac{e^{\alpha X}}{\alpha^2}, \qquad X < 0 \tag{26}$$

Now M has two values[7] M_1 and M_2 accordingly as $P_r > 2$ and $P_r < 2$. These values are

$$M_1 = \frac{\log[\{1+\sqrt{1-(2/P_r)}\}/\sqrt{(2/P_r)}]}{P_r\sqrt{1-(2/P_r)}} \tag{27}$$

$$M_2 = \frac{\cos^{-1}[\sqrt{P_r/2}]}{P_r\sqrt{(2/P_r)-1}} \tag{28}$$

The non-dimensional quantities are :

$$\Theta_1 = \left[\frac{K_2 T_1}{Q_0 L^2}\right]$$

$$\Theta_2 = \left[\frac{K_2 T_2}{Q_0 L^2}\right]$$

$$X = \left[\frac{x}{L}\right]$$

$$F_0 = \left[\frac{k_2 t}{L^2}\right]$$

$$\alpha = mL$$

$$\bar{K} = \left[\frac{K_1}{K_2}\right]$$

$$\sigma^2 = \left[\frac{k_1}{k_2}\right]$$

and

$$A = \left[\frac{\mu u_0^2}{Q_0 L^2}\right]$$

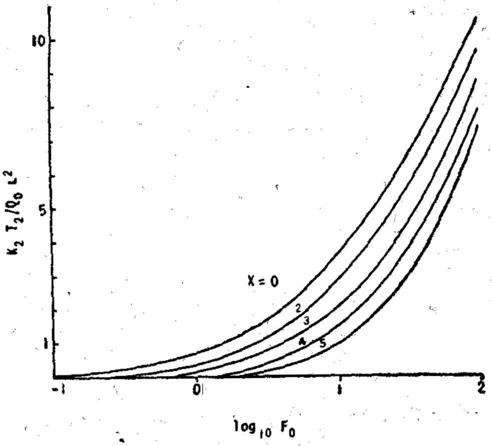

Fig. 1—Θ_2 for various depths (X) and λ (F_0).

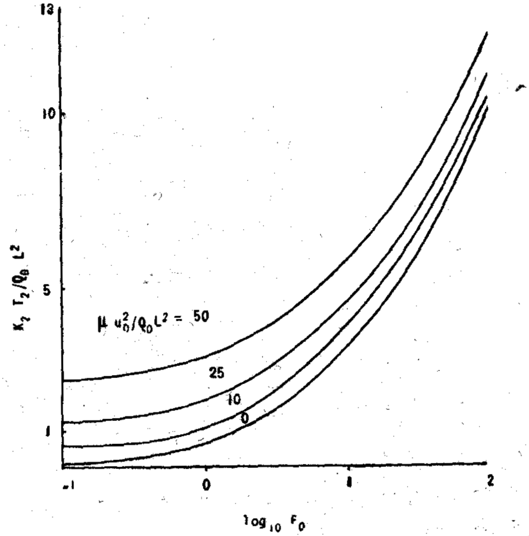

Fig. 2—The effect of A on Θ_2 at $X = 0$.

NUMERICAL COMPUTATION

Numerical computation for the solid region is carried out with the values of the parameters $\alpha = 1$, $\bar{K} = 0 \cdot 1$, $\sigma^2 \doteq 40$, $P_r = 0 \cdot 7$. The results are presented in Fig. 1 and Fig. 2 for various values of X, F_0 and A. The numbers on the curves denote the values of the parameter. In Fig. 1 the values of the non-dimensional temperature Θ_2 are plotted against $\text{Log}_{10} F_0$ for X=1, 2, 3, 4 and 5. Fig. 2 gives the effect of A on the non-dimensional temperature at the solid fluid interface. It is easily seen that A=0 corresponds to zero velocity (u_0=0) so that the problem reduces to the case of a composite medium consisting of fluid (at rest) and solid for which the solution can be obtained by putting A=0 in the expressions for Θ_1 and Θ_2. It is also observed that the non-dimensional temperature at the solid fluid interface increases with A.

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