AN ELASTIC INFINITE PLATE HAVING A HOLE RESEMBLING A FOUR-CUSPED HYPOTROCHOID UNDER UNIFORM PARTIAL LOADING

AQEEL AHMED

School of Studies in Mathematics and Statistics, Vikram University, Ujjain

(*Received* 17 *October* 1968 ; *revised* 7 *April* 1969)

The problem of an infinite plate perforated with a square hole with rounded corners has been solved when a part of the boundary is under uniform pressure and the rest is stress free. The solution has been obtained in a closed form by using Muskhelishvili's methods involving Cauchy's integrals. As a particular case, the solution has been obtained for the plate with the given hole under uniform pressure.

The problems of finite and infinite elastic regions, when a part of the boundary is subject to stresses and the rest is stress free are of theoretical interest. The problem of an elastic plate with a circular boundary was solved by Goodier[1]. Later on, the problem of an infinite plate with an elliptic hole was solved by mapping the elliptic hole conformally on the circular hole[2]. The latter method could be applied to other types of curvilinear boundaries.

In this paper, the problem of an infinite plate having a square hole with rounded corners has been solved by using conformal mapping. The plate is assumed stress free at infinity. A part of the boundary is subject to uniform pressure while the rest is stress free. As a particular case, the solution has been obtained for the plate with the given hole under uniform pressure.

STATEMENT OF THE PROBLEM

We consider the small deformation of an infinite plate consisting of a homogeneous isotropic, elastic material perforated with a square hole with rounded corners. The boundary of the plate is partially loaded with a uniform pressure while the plate is assumed stress free at infinity. Let the boundary of the plate be denoted by C. The transformation which maps the region exterior to C in z-plane on to that exterior to unit circle Γ in ζ-plane is

$$z = m(\zeta) = R\left(\zeta - \frac{K}{3\,\zeta^3}\right), \qquad \zeta = \rho e^{i\theta} \tag{1}$$

$$R > 0,\ 0 \leqslant K \leqslant 1$$

where R and K are real constants.

The curve C is a curve with rounded corners occuring at Fig. 1.

$$\theta = \frac{\pi}{4}\,,\ \frac{\pi}{4} + \frac{\pi}{2}\,,\ \frac{\pi}{4} + \pi,\ \frac{\pi}{4} + \frac{3\pi}{2}$$

we assume that the portion L' of the boundary C, the ends of which correspond to

$$\theta = \theta_1\,, \qquad \theta = \theta_2$$

is subject to a uniform pressure of magnitude P and the remaining portion L'' is stress free. Let the ends of the portion L' be denoted by z_1 and z_2 and the ends of the corresponding portions y' on Γ are denoted by σ_1 and σ_2 respectively (Fig. 2).

The values of z_1 and z_2 are given as

$$z_1 = R\left(\sigma_1 - \frac{K}{3\,\sigma_1^3}\right), \qquad z_2 = R\left(\sigma_2 - \frac{K}{3\,\sigma_2^3}\right)$$

FUNDAMENTAL FORMULAE

We use the generalised plane stress and quote the following standard formula[3]

$$\sigma_r + \sigma_\theta = 2\left[W(\zeta) + \bar{W}(\bar{\zeta}) \right] \tag{2}$$

$$\sigma_\theta - \sigma_r + 2i\,\tau_{r\theta} = 2\,\frac{\zeta^2}{\rho^2\,\bar{m}'(\bar{\zeta})}\left[\bar{m}(\bar{\zeta})\,W'(\zeta) + m'(\zeta)\,w(\zeta)\right] \tag{3}$$

$$2\mu\,(u + iv) = \chi\,\varphi(\zeta) - \frac{m(\zeta)}{\bar{m}'(\bar{\zeta})}\,\bar{\varphi}'(\bar{\zeta}) - \bar{\psi}(\bar{\zeta}) \tag{4}$$

where $\chi = 3 - \nu/1 + \nu$, μ and ν are the rigidity moduli and the Poisson's ratio of the material.

The potential functions are

$$\varphi(z) = \varphi[m(\zeta)] = \varphi(\zeta)$$
$$\psi(z) = \psi[m(\zeta)] = \psi(\zeta)$$

where

$$\varphi'(\zeta) = m'(\zeta)\,W(\zeta), \qquad \psi'(\zeta) = m'(\zeta)\,w(\zeta)$$

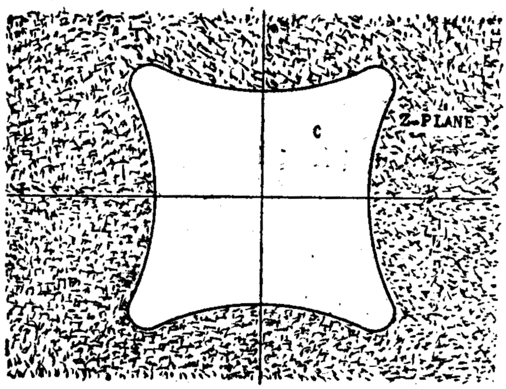

Fig. 1—The boundary C of an infinite plate perforated with a square hole having rounded corners.

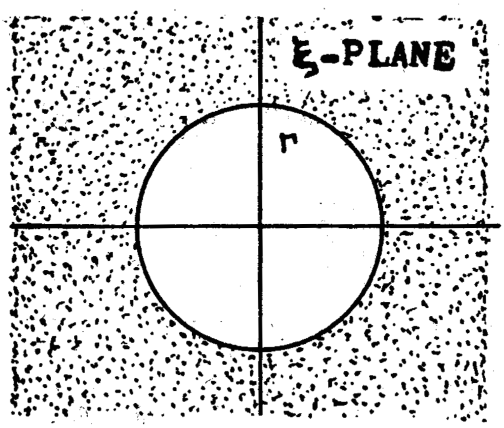

Fig. 2—The boundary described by a unit circle Γ in an infinite elastic plate.

GENERAL THEORY

Following Muskhelishvili[3], if the force across an arc element dl of the curve C is given by $(X_n+iY_n)\,dl$ and the stress in the material on the left of the curve (described with increasing l) is derived from $\varphi(z)$ and $\psi(z)$, then the boundary value of these functions on C satisfy

$$\varphi(z) + z\,\bar{\varphi}'(\bar{z}) + \bar{\psi}(\bar{z}) = i\int_0^l (X_n + i\,Y_n)\,dl = f_1 \text{ on } C$$

or in the transformed plane

$$\varphi(\sigma) + \frac{m(\sigma)}{\bar{m}'(\bar{\sigma})}\,\bar{\varphi}'(\bar{\sigma}) + \bar{\psi}(\bar{\sigma}) = f \qquad \sigma = e^{i\theta} \text{ on } \Gamma \tag{5}$$

where f is the value of f_1 in terms of σ.

The plate is free of stresses at infinity. Therefore the potential functions satisfying (5) are given as[3]:

$$\left.\begin{aligned}\varphi(\zeta) &= -\frac{X+iY}{2\pi(1+\chi)}\log\zeta + \varphi_0(\zeta),\\ \psi(\zeta) &= \frac{\chi(X-iY)}{2\pi(1+\chi)}\log\zeta + \psi_0(\zeta)\end{aligned}\right\} \tag{6}$$

where $\varphi_0(\zeta)$ and $\psi_0(\zeta)$ are holomorphic for $|\zeta| > |$ and where one can assume without any effect on stress distribution that

$$\varphi_0(\infty) = 0$$

Substituting these expressions in (5), it is seen that $\varphi_0(\zeta)$ and $\psi_0(\zeta)$ satisfy the same boundary condition (5) with the difference that f must be replaced by f_0 where

$$f_0 = f + \frac{X+iY}{2\pi}\log\sigma + \frac{X-iY}{2\pi(1+\chi)} \times \frac{\sigma^4 - K/3}{\sigma^2(1+K\sigma^4)} \tag{7}$$

Thus the functions $\varphi_0(\zeta)$ and $\psi_0(\zeta)$ can be easily determined from the equation

$$\varphi_0(\sigma) + \frac{m(\sigma)}{\bar{m}'(\bar{\sigma})}\,\bar{\varphi}'_0(\bar{\sigma}) + \bar{\psi}_0(\bar{\sigma}) = f_0 \tag{8}$$

Multiplying (8) by $\frac{1}{2\pi i}\, d\sigma/\sigma - \zeta$ and integrating, we get

$$\varphi_0(\zeta) = -\frac{1}{2\pi i}\int_\Gamma \frac{f_0\,d\sigma}{\sigma-\zeta} \tag{9}$$

Again taking the complex conjugate of (8) and multiplying by $\frac{1}{2\pi i}\, d\sigma/\sigma - \zeta$ and integrating, we get

$$\psi_0(\zeta) = -\frac{1}{2\pi i}\int_\Gamma \frac{\bar{f}_0\,d\sigma}{\sigma-\zeta} - \frac{\zeta^3\left(1 - \frac{K}{3}\zeta^4\right)}{\zeta^4+K}\,\varphi'_0(\zeta) \tag{10}$$

Hence $\varphi(\zeta)$ and $\psi(\zeta)$ are obtained with the help of (6), (9) and (10).

DETERMINATION OF STRESS POTENTIALS

In the present problem, the portion L' of the boundary is subject to uniform pressure of magnitude P and the rest part L'' is stress free. Therefore

$$X_n = -P\cos(\eta, x)$$
$$Y\eta = -P\cos(\eta, y)$$

and hence

$$(X_n + i\,Y_n)\,dl = -P\,(dy - i\,dx) = i\,P\,dz$$

Therefore

$$f = i\int_0^l (X_n + i\,Y_n)\,dl = -Pz = -P\,R\left(\sigma - \frac{K}{3\,\sigma^3}\right)\text{on } y' \qquad (11)$$

$$f = -Pz_2 \text{ on } y'' \qquad (12)$$

Moving round the contour in anticlockwise direction and coming back to z_1, the expression for f undergoes an increase

$$-P\,(z_2 - z_1) = P\,(z_1 - z_2)$$

The same increase will result for any subsequent circuit. Therefore the expression (7) takes the value

$$f_0 = f - \frac{P\,(z_1 - z_2)}{2\,\pi\,i}\log\sigma + \frac{P\,(\bar{z}_1 - \bar{z}_2)}{2\,\pi\,i\,(1+\chi)} - \frac{\sigma^4 - K/3}{\sigma^2\,(1 + K\,\sigma^4)} \qquad (13)$$

The value of the multivalued function $\log\sigma$ may be fixed arbitrarily at any point (*e.g.* at the point $\sigma_1 = e^{i\theta_1}$ corresponding to the point z_1); for a circuit round Γ the function $\log\sigma$ must vary continuously, so that $\log\sigma$ undergoes an increase $2\,\pi\,i$ and f_0 reverts to its original value. Hence f_0 will be single valued and continuous on the entire contour. Therefore putting this value of f_0 from (13) in (9) and integrating as explained elsewhere[2], we get

$$\varphi_0(\zeta) = \frac{P}{2\,\pi\,i}\left[\frac{K\,R}{3\,\zeta^3}\log\frac{\sigma_2}{\sigma_1} + \left\{R\left(\zeta - \frac{K}{3\,\zeta^3}\right) - z_2\right\}\log(\sigma_2 - \zeta)\right.$$
$$-\left\{R\left(\zeta - \frac{K}{3\,\zeta^3}\right) - z_1\right\}\log(\sigma_1 - \zeta) - (z_1 - z_2)\log\zeta$$
$$\left.-\frac{(\bar{z}_1 - \bar{z}_2)}{(1+\chi)}\cdot\frac{K}{3\,\zeta^2} - \frac{K}{\zeta}\left(\frac{1}{\sigma_2^{\,4}} - \frac{1}{\sigma_1^{\,4}}\right) - \frac{2\,K}{3\,\zeta^3}\left(\frac{1}{\sigma_2^{\,3}} - \frac{1}{\sigma_1^{\,3}}\right)\right] \qquad (14)$$

The function $\psi_0(\zeta)$ can be similarly found out. Thus by taking the complex conjugate of (7), then putting in (10) and integrating, we get

$$\psi_0(\zeta) = \frac{P}{2\,\pi\,i}\left[-\frac{R}{\zeta}\log\frac{\sigma_2}{\sigma_1} + \left\{R\left(\frac{1}{\zeta} - \frac{K}{3}\,\zeta^3\right) - \bar{z}_2\right\}\log(\sigma_2 - \zeta)\right.$$
$$-\left\{R\left(\frac{1}{\zeta} - \frac{K}{3}\,\zeta^3\right) - \bar{z}_1\right\}\log(\sigma_1 - \zeta) - (\bar{z}_1 - \bar{z}_2)\log\zeta$$

$$+ \frac{(z_1 - z_2)}{1 + \chi} \cdot \zeta^2 \frac{1 + K^2/3}{\zeta^4 + K} - \frac{K R}{3} \left\{ \tfrac{1}{3} (\sigma_2^3 - \sigma_1^3) + \tfrac{1}{2} (\sigma_2^2 - \sigma_1^2) \zeta \right.$$

$$\left. + (\sigma_2 - \sigma_1) \zeta^2 \right\} \Bigg] - \frac{\left(1 - \frac{K}{3} \zeta^4\right)}{\zeta^4 + K} \cdot \varphi_0' (\zeta) \tag{15}$$

The expressions (14) and (15) give the required stress potentials. Thus $\varphi(\zeta)$ and $\psi(\zeta)$ can be found out with the help of (6), (14) and (15).

DISPLACEMENT COMPONENTS ON THE BOUNDARY

The components of stresses and displacements can be found out with help of (2), (3) and (4). We give here the displacement components on the boundary of the hole. Thus the displacement u is given by

$$\frac{4 \pi \mu u}{P R} = \frac{K \chi}{3} \left(\sin 4 \theta_2 - \sin 4 \theta_1 \right) - \frac{K}{9} \left(\sin 3 \theta_2 - \sin 3 \theta_1 \right)$$

$$+ (\chi + 1) \left[\cos \theta_2 - \cos \theta_1 - \frac{K}{3} \left(\cos 3 \theta_2 - \cos 3 \theta_1 \right) \right] \theta$$

$$+ \frac{1 + \chi}{2} \left[\left(\frac{K}{3} \cos 3 \theta_2 - \cos \theta_2 \right) \left(\theta + \theta_2 - \pi \right) - \left(\frac{K}{3} \cos 3 \theta_1 - \cos \theta_1 \right) \left(\theta + \theta_1 + \pi \right) \right]$$

$$- \left[\left(\sin \theta_2 + \frac{K}{3} \sin 3 \theta_2 \right) \log \left(\left| 2 \sin \frac{\theta_2 - \theta}{2} \right| \right) - \left(\sin \theta_1 + \frac{K}{3} \sin 3 \theta_1 \right) \right.$$

$$\left. \times \log \left(\left| 2 \sin \frac{\theta_1 - \theta}{2} \right| \right) \right]$$

$$+ \left[\left(\theta_1 - \theta_2 \right) - \frac{K}{6} \left(\sin 2 \theta_2 - \sin 2 \theta_1 \right) - \tfrac{1}{2} \left(1 + \chi \right) \left(2 \pi + \theta_1 - \theta_2 \right) \right] \cos \theta$$

$$+ \left[\frac{K}{3} \left(\sin \theta_1 - \sin \theta_2 \right) - \frac{2 K \chi}{3} \left(\sin 3 \theta_1 - \sin 3 \theta_2 \right) + \left\{ \left(\sin \theta_1 - \sin \theta_2 \right) \right. \right.$$

$$\left. \left. + \frac{K}{3} \left(\sin 3 \theta_1 - \sin 3 \theta_2 \right) \right\} \times \frac{1}{3 (1 + \chi)} \left\{ K \chi + \frac{(3 + K^2)(1 + K)}{1 + K^2 + 2 K \cos 4 \theta} \right\} \right] \cos 2\theta$$

$$+ \left[\frac{K \chi}{3} (\theta_2 - \theta_1) + \frac{K}{6} (1 + \chi)(2 \pi + \theta_1 - \theta_2) \right] \cos 3\theta$$

$$+ \left[\frac{K}{6} \left(\cos 2 \theta_1 - \cos 2 \theta_2 \right) + \log \left(\left| \frac{\sin \frac{\theta_2 - \theta}{2}}{\sin \frac{\theta_1 - \theta}{2}} \right| \right) \right] \sin \theta$$

$$+ \left[\frac{K}{3} \left(\cos \theta_1 - \cos \theta_2 \right) - \frac{2 K \chi}{3} \left(\cos 3 \theta_1 - \cos 3 \theta_2 \right) + \frac{1}{3 (1 + \chi)} \left\{ \cos \theta_1 \right. \right.$$

$$\left. \left. - \cos \theta_2 - \frac{K}{3} \left(\cos 3 \theta_1 - \cos 3 \theta_2 \right) \right\} \times \left\{ K \chi - \frac{(3 + K^2)(1 - K)}{1 + K^2 + 2 K \cos 4 \theta} \right\} \right] \sin 2 \theta$$

$$+ \frac{K}{3} \log \left(\left| \frac{\sin \frac{\theta_2 - \theta}{2}}{\sin \frac{\theta_1 - \theta}{2}} \right| \right) \sin 3\theta \tag{16}$$

The displacement v is given by

$$\frac{4\pi\mu v}{PR} = \frac{K\chi}{3}\left(\cos 4\theta_2 - \cos 4\theta_1\right) - \frac{K}{9}\left(\cos 3\theta_2 - \cos 3\theta_1\right)$$

$$+\left(1+\chi\right)\left[\sin\theta_2 - \sin\theta_1 + \frac{K}{3}\left(\sin 3\theta_2 - \sin 3\theta_1\right)\right]\theta$$

$$-\left[\left(\frac{K}{3}\cos 3\theta_2 - \cos\theta_2\right)\left\{(\chi-1)\log\left(\left|2\sin\frac{\theta_2-\theta}{2}\right|\right) - \left(\theta_2+\theta-\pi\right)\right\}\right]$$

$$-\left(\frac{K}{3}\cos 3\theta_1 - \cos\theta_1\right)\left\{(\chi-1)\log\left(\left|2\sin\frac{\theta_1-\theta}{2}\right|\right) - (\theta_1+\theta+\pi)\right\}\Bigg]$$

$$-\tfrac{1}{2}\left[\left(\sin\theta_2 + \frac{K}{3}\sin 3\theta_2\right)\left(\theta_2+\theta-\pi\right) - \left(\sin\theta_1 + \frac{K}{3}\sin 3\theta_1\right)\left(\theta_1+\theta+\pi\right)\right]$$

$$-\left[\frac{K}{6}\left(\cos 2\theta_2 - \cos 2\theta_1\right) + (\chi-1)\log\left(\left|\frac{\sin\frac{\theta_2-\theta}{2}}{\sin\frac{\theta_1-\theta}{2}}\right|\right) + \left(2\pi+\theta_1-\theta_2\right)\right]\cos\theta$$

$$-\left[\frac{K}{3}(\cos\theta_2 - \cos\theta_1) - \frac{2K\chi}{3}(\cos 3\theta_2 - \cos 3\theta_1) + \frac{1}{3(1+\chi)}\left\{\cos\theta_2 - \cos\theta_1\right.\right.$$

$$\left.\left.-\frac{K}{3}(\cos 3\theta_2 - \cos 3\theta_1)\right\}\left\{K\chi + \frac{(3+K^2)(1+K)}{1+K^2+2K\cos 4\theta}\right\}\right]\cos 2\theta$$

$$+\frac{K}{3}\left[(\chi-1)\log\left(\left|\frac{\sin\frac{\theta_2-\theta}{2}}{\sin\frac{\theta_1-\theta}{2}}\right|\right) - (2\pi+\theta_1-\theta_2)\right]\cos 3\theta$$

$$-\left[(\theta_2-\theta_1) - \frac{K}{6}(\sin 2\theta_2 - \sin 2\theta_1) + \pi + \frac{\theta_1-\theta_2}{2}\right]\sin\theta$$

$$-\left[\frac{K}{3}(\sin\theta_1 - \sin\theta_2) - \frac{2K\chi}{3}(\sin 3\theta_1 - \sin 3\theta_2) + \frac{1}{3(1+\chi)}\left\{\sin\theta_1\right.\right.$$

$$\left.\left.-\sin\theta_2 + \frac{K}{3}(\sin 3\theta_1 - \sin 3\theta_2)\right\}\left\{K\chi - \frac{(3+K^2)(1-K)}{1+K^2+2K\cos 4\theta}\right\}\right]\sin 2\theta$$

$$+\left[\frac{K\chi}{3}(\theta_2-\theta_1) + \frac{K}{6}(2\pi+\theta_1-\theta_2)\right]\sin 3\theta \qquad (17)$$

The expressions (16) and (17) give the displacement components on the boundary of the hole except at the end points $\theta = \theta_1$ and $\theta = \theta_2$, where these become infinite. In fact these are the points yielding elastic failure. However, if one side of the square hole between $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = \frac{3\pi}{4}$ is loaded, then the displacement-components (16) and (17) reduce to the following expressions

$$\frac{4\pi\mu u}{PR} = -(1+\chi)\sqrt{2}\left(1+\frac{K}{3}\right)\theta + \frac{(1+\chi)}{2\sqrt{2}}\left(1+\frac{K}{3}\right)\left(\pi+2\theta\right)$$

$$-\frac{1}{\sqrt{2}}\left(1+\frac{K}{3}\right)\log\left(\left|\frac{\sin\left(\frac{3\pi}{8}-\frac{\theta}{2}\right)}{\sin\left(\frac{\pi}{8}-\frac{\theta}{2}\right)}\right|\right) + \left(\frac{K}{3}-\frac{\pi}{4}(3\chi+5)\right)\cos\theta$$

$$+\frac{K\pi}{12}(3\chi+1)\cos 3\theta + \log\left(\left|\frac{\sin\left(\frac{3\pi}{8}-\frac{\theta}{2}\right)}{\sin\left(\frac{\pi}{8}-\frac{\theta}{2}\right)}\right|\right)\sin\theta$$

$$\sqrt{2}\left[\frac{K}{3}(1+2\chi) + \frac{1}{3(1+\chi)}\left(1+\frac{K}{3}\right)\left\{K\chi - \frac{(3+K^2,(1-K)}{1+K^2+2K\cos 4\theta}\right\}\right]\sin 2\theta$$

$$+\frac{K}{3}\log\left(\left|\frac{\sin\left(\frac{3\pi}{8}-\frac{\theta}{2}\right)}{\sin\left(\frac{\pi}{8}-\frac{\theta}{2}\right)}\right|\right)\sin 3\theta \qquad (18)$$

$$\frac{4\pi\mu v}{PR} = -\frac{K}{9}(6\chi+\sqrt{2}) + \frac{1}{\sqrt{2}}\left(1+\frac{K}{3}\right)\Bigg[(1-\chi)$$

$$\log\left(\left|4\sin\left(\frac{3\pi}{8}-\frac{\theta}{2}\right)\sin\left(\frac{\pi}{8}-\frac{\theta}{2}\right)\right|\right) + (\pi+2\theta)\Bigg]$$

$$+\left[(1-\chi)\log\left(\left|\frac{\sin\left(\frac{3\pi}{8}-\frac{\theta}{2}\right)}{\sin\left(\frac{\pi}{8}-\frac{\theta}{2}\right)}\right|\right) + \frac{3\pi}{2}\right]\cos\theta$$

$$+\sqrt{2}\left[\frac{K}{3}(1+2\chi) + \frac{1}{3(1+\chi)}\left(1+\frac{K}{3}\right)\left\{K\chi + \frac{(3+K^2)(1+K)}{1+K^2+2K\cos 4\theta}\right\}\right]\cos 2\theta$$

$$-\frac{K}{3}\left[(1-\chi)\log\left(\left|\frac{\sin\left(\frac{3\pi}{8}-\frac{\theta}{2}\right)}{\sin\left(\frac{\pi}{8}-\frac{\theta}{2}\right)}\right|\right) - \frac{3\pi}{2}\right]\cos 3\theta$$

$$+\left(\frac{K}{3}-\frac{\pi}{4}\right)\sin\theta + K\pi\left(\frac{\chi}{6}-\frac{\pi}{4}\right)\sin 3\theta \qquad (19)$$

The expressions (18) and (19) give the displacements at any point on the boundary, when one side of the hole lying between $\theta_1 = \frac{\pi}{4}$; $\theta_2 = \frac{3\pi}{4}$ is loaded uniformly.

SOLUTION FOR THE PLATE UNDER ALL ROUND UNIFORM PRESSURE

The solution of the problem when the plate is under all round uniform pressure may be easily deduced from the above results by putting

$$\theta_2 = 2\pi + \theta_1 \quad , \quad \sigma_1 = \sigma_2 \quad \text{and} \log \frac{\sigma_2}{\sigma_1} = 2\pi i$$

The resultant force vector of the applied forces on the boundary is also zero i.e.,

$$X + iY = 0$$

Thus the stress potentials (14) and (15) reduce to

$$\varphi(\zeta) = \frac{PKR}{3\zeta^3} \tag{20}$$

$$\psi(\zeta) = -\frac{PR}{\zeta} + \frac{PRK\left(1 - \frac{K}{3}\zeta^4\right)}{\zeta(\zeta^4 + K)} \tag{21}$$

The displacement components (16) and (17) take values

$$\frac{4\pi\mu u}{PR} = -2\pi\cos\theta + \tfrac{2}{3}\pi K\chi\cos 3\theta \tag{22}$$

$$\frac{4\pi\mu v}{PR} = -2\pi\sin\theta - \tfrac{2}{3}\pi K\chi\sin 3\theta \tag{23}$$

If we put $K = 0$ in (20) to (23), we get the known result for the plate with a circular hole[2].

ACKNOWLEDGEMENT

I am very much thankful to Prof. G. Paria, D. Sc., of Shri Govindram Saksaria Technological Institute, Indore for his kind help and guidance in the preparation of the paper.

REFERENCES

1. GOODIER, J. N., *J. Appl. Mech.*, 54 (1952), 173.
2. MUSKHELISHVILI, *ibid.*, 54 (1952), 173.
3. MUSKHELISHVILI, N. I., "Some Basic Problems in the Mathematical Theory of Elasticity", (P. Noordhoff, Holand) 1953, pp 180-185.