STABILITY OF VISCOUS FLOW BETWEEN TWO CONCENTRIC ROTATING POROUS CYLINDERS

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The stability of a viscous flow between two concentric rotating porous cylinders has been examined when the difference in radii of the cylinders is small in comparison with their mean radius. Critical Taylor numbers have been calculated for various wave numbers and velocity ratios of the cylinders. Theoretical results show that injection at the outer cylinder improves the stability of the flow whereas suction has an opposite effect.

The problem of stability of viscous flow between two rotating concentric cylinders has been the subject of theoretical and experimental investigations by Taylor[1], Meksyn[2], Chandrasekhar[3] and Sparrow, et al[4]. Taylor[1] was first to study the problem by considering the narrow gap between the cylinders and obtained a solution by expanding the velocities in terms of a series of Bessel functions of order zero and unity. Meksyn[2] derived a closed form solution applicable only to the narrow gap for the instability condition by using the method of asymptotic expansion of the velocities in inverse powers of Taylor number, taken to be a large parameter. Sparrow et al[4] studied the stability of flow between the rotating coaxial cylinders having an arbitrary gap. Their results are used in defining the range of applicability of the solutions of Taylor and Meksyn which were derived only for narrow gap conditions. Chandrasekhar[3] has also considered several aspects of the stability problems for the narrow gap. Mention may also be made of a recent paper by Richie[5] who has analysed the stability of viscous flow between the eccentric infinite cylinders having a narrow gap. An approximate solution of the resulting eigen value problem has been found by rotating the inner cylinder and keeping the outer one stationary. The difference in radii of the cylinders has been taken to be small. It has been established that increase in the eccentricity ratio has a destabilising effect. Thomas & Walters[6] have also drawn a similar conclusion to illustrate the effect of elasticity on the stability of flow.

This paper is an attempt to investigate the stability of the flow between the concentric rotating porous cylinders in the presence of radial velocity. In the physical sense the radial velocity will mean suction or injection according as it is directed away from or towards the axis. The study is restricted to the case in which the difference in radii of the cylinders is small in comparison with their mean radius.

Critical Taylor numbers depicting the onset of instability have been calculated for various wave numbers and velocity ratios of the cylinders. The theoretical results show that suction has a destabilising effect on the flow while injection tends to stabilise it.

FUNDAMENTAL EQUATIONS

The axisymmetric flow of a viscous incompressible fluid contained between two rotating coaxial cylinders is governed by the equations

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} = - \frac{\partial}{\partial r}\left(\frac{p}{\rho}\right) + \nu \left(\nabla^2 u_r - \frac{u_r}{r^2}\right) \tag{1}$$

$$\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} = \nu \left(\nabla^2 u_\theta - \frac{u_\theta}{r^2}\right) \tag{2}$$

$$\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} = - \frac{\partial}{\partial z}\left(\frac{p}{\rho}\right) + \nu \nabla^2 u_z \tag{3}$$

where
$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

The equation of continuity is

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0 \tag{4}$$

u_r, u_θ and u_z are components of velocity in the increasing r, θ and z directions, ρ the density of the fluid, p the pressure and ν the kinematic viscosity. z-axis coincides with the axis of the coaxial cylinders.

In the presence of suction (or injection) at the walls of the cylinders, these equations admit a stationary solution of the form

$$\left.\begin{aligned} u_r &= R_1 u_1/r = U(r) \\ u_\theta &= A r^{\lambda+1} + B/r = V(r) \\ u_z &= 0 \end{aligned}\right\} \tag{5}$$

where $\lambda = \dfrac{R_1 u_1}{\nu}$, u_1 the radial velocity of the fluid at $r = R_1$ (radius of the inner cylinder, A and B are the arbitrary constants. If Ω be the angular velocity of the fluid, we have

$$\Omega = A r^{\lambda} + B/r^2 \tag{6}$$

If the inner cylinder $r=R_1$ and outer cylinder $r=R_2$ rotate with angular velocities Ω_1 and Ω_2 respectively we have

$$\left.\begin{aligned} u_\theta &= R_1 \Omega_1 \quad \text{for } r = R_1 \\ u_\theta &= R_2 \Omega_2 \quad \text{for } r = R_2 \end{aligned}\right\} \tag{7}$$

Substituting in (6), the constants A and B are

$$\left.\begin{aligned} A &= -\Omega_1 \eta^2 \frac{1-\mu/\eta^2}{R_2^{\lambda}\left(1-\eta^{\lambda+2}\right)} \\ B &= R_1^2 \Omega_1 \frac{\left(1-\mu\eta^{\lambda}\right)}{\left(1-\eta^{\lambda+2}\right)} \end{aligned}\right\} \tag{8}$$

where $\mu = \Omega_2/\Omega_1$ and $\eta = R_1/R_2$

Rayleigh's Criterion

Rayleigh's Criterion for stability of inviscid Couette flow requires

$$\Phi(r) = \frac{2}{r}\,\Omega\,\frac{d}{dr}\,(r^2\,\Omega_1) > 0 \tag{9}$$

Substituting the value of Ω from (6) in (9) we get

$$\Phi = 2(\lambda+2)\,AB\left(r^{\lambda-2} + \frac{A}{B}\,r^{2\lambda}\right) \tag{10}$$

Putting the values of A and B from (8) in (10) Φ can be expressed as

$$\Phi = \frac{-2(\lambda+2)\,\Omega_1^2\,\eta^4\left(1-\mu\eta^{\lambda}\right)(1-\mu/\eta^2)}{\left(1-\eta^{\lambda+2}\right)^2}\left[r^{\lambda-2} - \frac{(1-\mu/\eta^2)\,r^{2\lambda}}{1-\mu\eta^{\lambda}}\right] \tag{11}$$

In the case of suction at the outerwall, Rayleigh's criterion for stability of an inviscid flow yields the condition $\mu > \eta^2$ which is same for all value of $\lambda \geqslant 0$. However when the fluid is injected into the flow field from the outer cylinder i.e., when $\lambda < 0$, the criterion requires an additional condition $-(\lambda+2) > 0$.

PERTURBATION EQUATIONS

Assuming perturbations to be axisymmetric and independent of θ, the linearised equations of the problem can be written as

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial U}{\partial r} + U\frac{\partial u_r}{\partial r} - \frac{2V}{r}u_\theta = -\frac{\partial \bar{w}}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2}\right) \tag{12}$$

$$\frac{\partial u_\theta}{\partial t} + \left(\frac{\partial V}{\partial r} + \frac{V}{r}\right)u_r + \left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\right)U = \nu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2}\right) \tag{13}$$

$$\frac{\partial u_z}{\partial t} + U\frac{\partial u_z}{\partial r} = -\frac{\partial \bar{w}}{\partial z} + \nu\nabla^2 u_z \tag{14}$$

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0 \tag{15}$$

where $U + u_r$, $V + u_\theta$, u_z are the velocity components in the perturbed state, and

$$\frac{\delta p}{\rho} = \bar{w}. \tag{16}$$

The disturbances can be analysed into normal modes of the form.

$$\left.\begin{aligned} u_r &= e^{pt}\,u(r)\cos kz \\ u_\theta &= e^{pt}\,v(r)\cos kz \\ u_z &= e^{pt}\,w(r)\sin kz \\ \bar{w} &= e^{pt}\,\bar{w}(r)\cos kz \end{aligned}\right\} \tag{17}$$

where k is a wave number of the disturbance and p is a constant.

Using (17), equations (12) to (15) become

$$\nu\,(DD_* - k^2 - p/\nu)\,u + \frac{2Vv}{r} - u\frac{\partial U}{\partial r} - U\frac{\partial u}{\partial r} = \frac{\partial \bar{w}}{\partial r} \tag{18}$$

$$\nu\,(DD_* - k^2 - p/\nu)\,v - (D_*V)u - (D_*v)\,U = 0 \tag{19}$$

$$\nu\,(D_*D - k^2 - p/\nu)\,w - U\frac{\partial w}{\partial r} = -\,k\bar{w} \tag{20}$$

$$D_*u = -kw \tag{21}$$

and
$$\nabla^2 = DD_* + \frac{1}{r^2} - k^2 = D_*D - k^2 .$$

where
$$D = \frac{d}{dr}\,, \qquad D_* = \frac{d}{dr} + \frac{1}{r}$$

Eliminating w from the above equations, they can be rearranged as:

$$\frac{\nu}{k^2}\,(DD_* - k^2 - p/\nu)\,(DD_* - k^2)\,u - \frac{1}{k^2}\,U\,\frac{d^2}{dr^2}\,(D_*u) - \frac{1}{k^2}\,\frac{dU}{dr}\,\frac{d}{dr}\,(D_*u)$$
$$+\,u\,\frac{dU}{dr} + U\,\frac{du}{dr} = \frac{2V}{r}\,v \tag{22}$$

$$\nu\,(DD_* - k^2 - p/\nu)\,v = (D_*V)\,u + (Dv_*)U \tag{23}$$

substituting $U = \dfrac{\lambda\nu}{r}$ and $V = A\,r^{\lambda+1} + B/r$ in (22) and (23) we get

$$\frac{\nu}{k^2}\,(DD_* - k^2 - p/\nu)\,(DD_* - k^2)\,u - \frac{\lambda\nu}{k^2}\,\frac{1}{r}\,\frac{d^2}{dr^2}\,(D_*u) + \frac{\lambda\nu}{k^2}\,\frac{1}{r^2}\,\frac{d}{dr}\,(D_*u)$$
$$-\,\frac{\lambda\nu u}{r^2} + \frac{\lambda\nu}{r}\,\frac{du}{dr} = 2v\,(Ar^{\lambda} + B/r^2) \tag{24}$$

and
$$\nu\,(DD_* - k^2 - p/\nu)\,v = Ar^{\lambda}\,(\lambda + 2)\,u + (D_*\,v)\,\frac{\lambda\nu}{r} \tag{25}$$

NARROW GAP APPROXIMATION

Applying narrow gap approximation i.e., considering the difference in the radii of the cylinders $R_2 - R_1$ to be small in comparison with the mean radius $\dfrac{R_1 + R_2}{2}$, we can replace D_* with D and further (8) and (6) may be approximated as

$$A\,r^{\lambda} = -\,\frac{\Omega_1}{\lambda + 2}\left[(1-\mu)\,(R_2/d) + \lambda\,(1-\mu)\,(\zeta - 1) - 2\right] \tag{26}$$

$$A\,r^{\lambda} + B/r^2 = \Omega_1\,[1 - (1-\mu)\,\zeta\,] \tag{27}$$

where
$$\zeta = \frac{r - R_1}{R_2 - R_1}$$

The equations (24) and (25) are made dimensionless by measuring radial distances in terms of $d = R_2 - R_1$. These can be rewritten in the light of narrow gap approximation as

$$(D^2 - a^2 - \sigma)(D^2 - a^2)\,u = \frac{2\Omega_1 d^2 a^2}{\nu}\,[1-(1-\mu)\,\xi\,]\,v \tag{28}$$

and

$$(D^2 - a^2 - \sigma)\,v = -\,\frac{\Omega_1 d^2 u}{\nu}\Big[(1-\mu)(R_2/d) + \lambda(1-\mu)(\zeta - 1) - 2\Big] \tag{29}$$

where $\qquad k = a/d \quad \text{and} \quad \sigma = pd^2/\nu$

When the marginal state is stationary, equations (28) and (29) are simplified further by putting $\sigma = 0$ and applying the transformation $u \longrightarrow \dfrac{2\Omega_1 d^2 a^2}{\nu}\,u$

$$(D^2 - a^2)^2\,u = (1 + \alpha\,\zeta)\,v \tag{30}$$

$$(D^2 - a^2)\,v = -\,Ta^2 u\,[(1 - k_1) + k_1\zeta] \tag{31}$$

where

$$T = -\,\frac{2\Omega_1^{\,2} d^4}{\nu^2}\,(\alpha\,R_2/d + 2) \tag{32}$$

and

$$k_1 = \frac{\alpha\lambda}{\alpha\,\dfrac{R_2}{d} + 2} \tag{33}$$

SOLUTION OF THE CHARACTERISTIC VALUE PROBLEM

Following Chandrasekhar's technique we may take

$$v = \sum_{m=1}^{\infty} C_m \sin m\pi\zeta \tag{34}$$

and because of (30) one has

$$u = \sum_{m=1}^{\infty} \frac{C_m}{(m^2\pi^2 + a^2)^2}\Big\{A_1^{(m)}\cosh a\zeta + B_1^{(m)}\sinh a\zeta + A_2^{(m)}\,\zeta\cosh a\zeta + B_2^{(m)}\,\zeta\sinh a\zeta + (1+\alpha\zeta)\sin m\pi\zeta + \frac{4\alpha m\pi}{m^2\pi^2 + a^2}\cos m\pi\zeta\Big\} \tag{35}$$

Under the boundary conditions $u = Du = 0$ at $\zeta = 0$ and $\zeta = 1$, the constants of integrations $A_1^{(m)}$, $A_2^{(m)}$, $B_1^{(m)}$, $B_2^{(m)}$, are determined by following Chandrasekhar[2].

Multiplying equation (31) by $\sin n\pi\zeta$ after substituting the values of u, v, $A_1^{(m)}$, $A_2^{(m)}$, $B_1^{(m)}$, and $B_2^{(m)}$ and integrating the resulting expression, we get a system of linear homogeneous equations having arbitrary constants $C_m/(m^2\pi^2 + a^2)^2$. Since these are not all zero, after some simplifications we must have

$$\Bigg\| \frac{4mn\pi^2 \alpha (1-k_1)}{(m^2\pi^2+a^2)(n^2\pi^2+a^2)} \left\{ (-1)^{m+n} - 1 \right\} - \frac{2amn\pi^2 (1-k_1)}{(n^2\pi^2+a^2)^2 (\sinh^2 a - a^2)} \times$$

$$\times \Big[(\sinh a \cosh a - a) \left\{ 1 + (1+\alpha)(-1)^{n+m} \right\} +$$

$$+ (\sinh a - a \cosh a) \left\{ (-1)^{n+1} + (1+\alpha)(-1)^{m+1} \right\} -$$

$$- \frac{4a\alpha \sinh a}{(m^2\pi^2+a^2)} \left\{ \sinh a + (-1)^{m-1} \right\} \left\{ (-1)^{m+n} - 1 \right\} \Big] +$$

$$+ \frac{4mn\pi^2 \alpha (-1)^{m+1}}{(m^2\pi^2+a^2)(n^2\pi^2+a^2)} k_1 - \frac{2an\pi (-1)^{n+1}}{(n^2\pi^2+a^2)^2} k_1 \Big[\frac{-4\alpha m\pi}{m^2\pi^2+a^2} \sinh a +$$

$$+ \frac{m\pi}{\triangle} \Big\{ 2 \sinh a - a \cosh a + \beta_m (\sinh a \cosh a + a \cosh^2 a - 2a \sinh^2 a) +$$

$$+ \gamma_m (\sinh a \cosh a - 2a) \Big\} + \frac{m\pi}{\triangle (-1)^{n+1}} \Big\{ a + \beta_m (\sinh a + a \cosh a) -$$

$$- \gamma_m \sinh a \Big\} \Big] + \frac{n\pi(6a^2 - 2n^2\pi^2)k_1}{(n^2\pi^2+a^2)^3} \Big[- \frac{m\pi}{\triangle} \Big\{ \sinh^2 a + \beta_m (a \sinh a + a^2 \cosh a) -$$

$$- a\gamma_m \sinh a \Big\} - \frac{m\pi (-1)^{n+1}}{\triangle} \Big\{ a \sinh a + \beta_m (a^2 + a \sinh a \cosh a) -$$

$$- \gamma_m \sinh^2 a \Big\} \Big] + \tfrac{1}{2} \delta_{mn} + X_{nm} - \tfrac{1}{2} \frac{(n^2\pi^2+a^2)^3}{a^2 T} \delta_{nm} \Bigg\| = 0 \qquad (36)$$

where

$$X_{nm} = \begin{cases} 0 \text{ if } m+n \text{ is even and } m \neq n \\ \frac{1}{4}\left(\alpha + k_1 - \frac{\alpha k_1}{3} - \frac{\alpha k_1}{n^2\pi^2} \right) \text{ if } m = n \\ \frac{4mn}{n^2 - m^2} \left[\frac{2\alpha}{m^2\pi^2 + a^2} - \frac{\alpha + k_1}{\pi^2(n^2 - m^2)} \right] \text{ if } m+n \text{ is odd.} \end{cases}$$

where

$$\triangle = \sinh^2 a - a^2$$

$$\beta_m = \frac{4\alpha}{m^2\pi^2 + a^2} \left\{ (-1)^{m+1} + \cosh a \right\}$$

and

$$\gamma_m = (-1)^{m+1} (1+\alpha) + \frac{4\alpha}{m^2\pi^2 + a^2} a \sinh a.$$

The first approximation of the equation (36) is obtained by putting (1, 1) element of the matrix equal to zero. Thus we have

$$\frac{(\pi^2+a^2)^3}{2a^2T} = \tfrac{1}{4}\left(\alpha + k_1 - \frac{\alpha k_1}{3} - \frac{\alpha k_1}{\pi^2}\right) + \tfrac{1}{2} - \frac{2a\pi^2\,(2+\alpha)\,(1-k_1)}{(\pi^2+a^2)^2\;(\sinh^2 a - a^2)} \times$$

$$\times\left\{(\sinh a \cosh a - a) + (\sinh a - a\cosh a)\right\} + \frac{4\pi^2\alpha k_1}{(\pi^2+a^2)^2} -$$

$$- \frac{2a\pi k_1}{(\pi^2+a^2)^2}\left[-\frac{4\pi\alpha}{\pi^2+a^2}\sinh a + \frac{\pi}{\sinh^2 a - a^2}\times\right.$$

$$\times\left\{-a\,(1+\cosh a+2\alpha) + \sinh a\,(1-\alpha) + (1+\alpha)\sinh a\cosh a\right\} +$$

$$+ \frac{4\pi\alpha}{(\pi^2+a^2)(\sinh^2 a - a^2)}\left\{2a + 2a\cosh a + \sinh a\,(1 - 2a^2 - a\sinh a +\right.$$

$$\left.\left. + 2\cosh a + \cosh^2 a)\right\}\right] + \frac{\pi^2\,(6a^2 - 2\pi^2)\,k_1}{(\pi^2+a^2)^3\;(\sinh^2 a - a^2)}\times$$

$$\times\left\{\alpha\,(a\sinh a + \sinh^2 a) - \frac{4\alpha}{\pi^2+a^2}\,(1+\cosh a)\,(2a\sinh a + 2a^2)\right\} \quad (37)$$

In the limiting case when there is no suction or injection of the fluid i.e., when λ or $k_1 \to 0$ equation (37) reduces to Chandrasekhar's result.

The values of $\log T_c$ for a fixed value of $a = 3\cdot 12$, are plotted against μ in Fig. 1. The various curves are presented for $\lambda \lesseqgtr 0$. The curve for $\lambda = 0$ gives the case of no radial velocity.

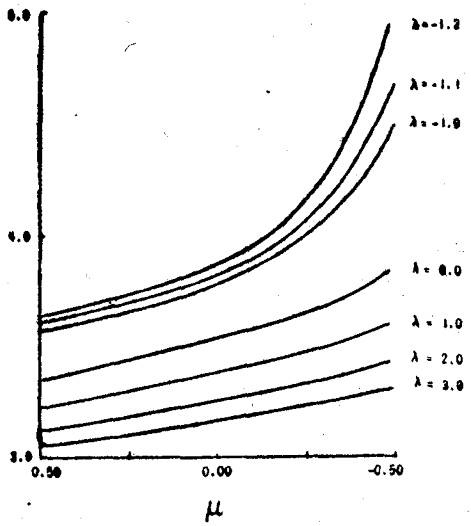

Fig. 1—Critical Taylor number T_c for the onset of instability as a function of μ for various values of λ and $a = 3\cdot 12$.

DISCUSSION

The numerical results have been restricted to the first approximation only. In the case of suction it is shown that the critical Taylor number T_c decreases steadily as the suction parameter λ increases. This implies that suction at the outer wall has a destabilising effect on the fluid confined between two coaxial rotating cylinders. This behaviour of the flow is similar to that which has been found by Thomas & Walters[6] while studying the effect of the increase in elastic properties of the fluid on the stability of visco-elastic flows. The effect of eccentricity studied by Richie[5] is also similar. In the case of injection through the outer cylinder, it is observed that the critical Taylor number increases steadily. It is concluded that injection tends to damp out the disturbances. Numerical results in this case have been calculated for $\lambda = -1, -1\cdot 1$ and $-1\cdot 2$. Negative values of λ will have to be less than 2 as we have seen in the stability criterion. d/R_2 has been taken as $0\cdot 1$ in both the cases. We find that the first approximation does not give the correct values of T_c for $\lambda >$ or < 0 in the region $0\cdot 5 < \mu < 1\cdot 0$.

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