VORTICITY IN DIABATIC STEADY GAS FLOWS

G. PURUSHOTHAM

Govt. Nagarjunasagar Engineering College, Hyderabad

MADHUSUDAN

Agarwal Evening Science College, Hyderabad

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Various kinematic and kinetic properties of steady diabatic gas flows are studied considering the geometry of triply orthogonal spatial curves of congruences one related to the vortexlines and the other to their principal normals and binormals.

Several physical and chemical phenomena invalidate the assumption of adiabatic flow in many compressible flow problems. The inviscid non-conducting steady gas flows with energy addition by heat sources are termed diabatic and corresponding to heating processes are thermodynamically reversible. The results from diabatic flow studies provide the basic insight into heat addition effects, which is necessary. Hicks[1] formulated the fundamental equations governing diabatic steady gas flows and attempted to study the geometry of plane flows. Subsequently the geometric properties of spatial diabatic gas flows were studied by us, considering the geometry of the streamline. The geometry of vortexline is not, however, correlated with physical problems. Consequently, considering triply orthogonal spatial curves of congruences formed by vortexlines, the principal normals and the binormals, we have studied in this paper the kinematic and kinetic properties of diabatic flows.

The basic conditions to be satisfied by the geometric parameters of triply orthogonal curves are obtained. Defining velocity vector, the magnitude of vorticity is determined and analytical conditions are obtained, which are more elegant than those of Suryanarayana[3]. The variation of flow parameters is studied at length.

FUNDAMENTAL EQUATIONS

The fundamental equations governing steady diabatic gas flow in the absence of extraneous forces, in Crocco's velocity vector field are given below in the usual notation[1].

$$\operatorname{div}\left\{\vec{w}\,(1-w^2)^{\frac{1}{\gamma-1}}\right\} = q\left(1+\frac{\gamma+1}{\gamma-1}w^2\right)\left(1-w^2\right)^{\frac{2-\gamma}{\gamma-1}} \tag{1}$$

$$\nabla \log p_t = \frac{2\gamma}{\gamma-1}\left(\frac{\vec{w}\wedge \operatorname{curl}\vec{w} - q\,\vec{w}}{1-w^2}\right) \tag{2}$$

$$V_t \vec{w} \wedge \text{curl } \vec{w} = C_p \nabla T_t - T \nabla S \tag{3}$$

$$p_t = p\,(1 - w^2)^{\frac{-\gamma}{\gamma - 1}} \tag{4}$$

where $\vec{w}$, q, γ, p_t, V_t, C_p, T_t, T and S are reduced velocity vector, the heat content, the adiabatic exponent, the total pressure, the limiting velocity, the stagnation enthalpy, the temperature and specific entropy respectively.

GEOMETRIC RELATIONS

Considering $\vec{t}$, $\vec{n}$ and $\vec{b}$ as triply orthogonal unit tangent vectors to the curves of congruences formed by the vortexlines, the principal normals and their binormals respectively and denoting $\frac{d}{ds}$, $\frac{d}{dn}$, $\frac{d}{db}$ as directional derivatives along these vectors, also selecting $\vec{r}$ as the position vector on a vortexline we have the following geometric results[4,5]:

$$\frac{d\vec{r}}{ds} = \vec{t} = \frac{\vec{\zeta}}{\zeta} \tag{5}$$

$$\frac{d\vec{t}}{ds} = \vec{n}\,k \quad (6) \qquad \frac{d\vec{b}}{ds} = -\vec{n}\,\tau \quad (7) \qquad \frac{d\vec{n}}{ds} = \vec{b}\,\tau - \vec{t}\,k \quad (8)$$

$$J = \text{div } \vec{t} = -(k' + k'') \quad (9) \qquad \text{div } \vec{n} = -k \quad (10) \qquad \text{div } \vec{b} = 0 \quad (11)$$

Using solenoidal property of rotational vectors of these unit vectors we obtain the following:

$$\frac{dk}{db} + \frac{d}{ds}(\sigma' - \sigma'') + (k' + k'')(\sigma'' - \sigma') = 0 \tag{12}$$

$$k\,(\tau + \sigma'') - \frac{dk'}{db} - \frac{d}{dn}(\sigma'' + \tau) = 0 \tag{13}$$

$$\frac{dk''}{db} - kk'' + \frac{d}{db}(\sigma' - \tau) = 0 \tag{14}$$

These constitute the basic conditions to be satisfied by triply orthogonal spatial curves of congruences. Here (k, k', k''), J and $(\tau, \sigma', \sigma'')$ are the curvatures, the mean curvature, and the torsions of the curves described.

KINEMATIC PROPERTIES OF FLOWS

We study here some of the kinematic and kinetic properties of the flows described, Let us define the velocity vector $\vec{w}$ as

$$\vec{w} = \vec{t}\, w_t + \vec{n}\, w_n + \vec{b}\, w_b \tag{15}$$

where w_t, w_n and w_b are the resolved parts of the velocity components along a vortexline, the principal normal and binormals respectively.

Operating curl on (15) and equating to $\vec{\zeta}$ we obtain

$$\zeta = (\sigma' - \sigma'')\, w_t + \frac{dw_b}{dn} - \frac{dw_n}{db} \tag{16}$$

$$\frac{dw_t}{db} - w_n\,(\tau + \sigma'') + k''\, w_b - \frac{dw_b}{ds} = 0 \tag{17}$$

$$kw_t - \frac{dw_t}{dn} - kw_n + \frac{dw_n}{ds} + (\sigma' - \tau)\, w_b = 0 \tag{18}$$

Relation (16) gives the magnitude of the vorticity and (17) & (18) the conditions to be satisfied by the velocity. These results are more elegant than those of Suryanarayana[3].

Using solenoidal property the vorticity, we obtain

$$k' + k'' = \frac{d}{ds} \log \zeta \tag{19}$$

If the normal congruences are minimal, the vorticity in magnitude shall remain uniform along an individual vortexline.

Making use of (15) in (1) we obtain the continuity equation as

$$\frac{dw_t}{ds} + \frac{dw_n}{dn} + \frac{dw_b}{db} - kw_n - (k' + k'')\, w_t$$

$$+ \left(w_t \frac{d}{ds} + w_n \frac{d}{dn} + w_b \frac{d}{db}\right) \log (1 - w^2)^{\frac{1}{\gamma - 1}}$$

$$= q\left(1 + \frac{\gamma + 1}{\gamma - 1}\, w^2\right) \div (1 - w^2) \tag{20}$$

This expresses the conservation of mass in a vortexline flow.

Forming the scalar products successively by $\vec{t}$, $\vec{n}$, $\vec{b}$ of (2) we obtain

$$\frac{dp_t}{ds} = - \frac{2\gamma q\, w_t\, p_t}{(\gamma - 1)(1 - w^2)} \tag{21}$$

$$\frac{dp_t}{dn} = \frac{2\gamma\, p_t\, (\zeta w_b - q w_n)}{(\gamma - 1)\,(1 - w^2)} \tag{22}$$

$$\frac{dp_t}{db} = -\, 2\gamma\, p_t\, (\zeta w_n + w_b q) \div (\gamma - 1)\,(1 - w^2) \tag{23}$$

From (21) we observe that the total pressure remains uniform along a vortexline if either the flow is complex-lamellar or the fluid is adiabatic. Also magnitude of the pressure decreases or increases in regions $w^2 \lessgtr 1$ for Chapygin gas. The adiabatic case can be discussed as a special case. For adiabatic gas (22) and (23) yield

$$w_n \frac{dp_t}{dn} + w_b \frac{dp_t}{db} = 0 \tag{24}$$

This shows that the total pressure remains uniform along a vortexline.

Taking scalar product of (3) by $\vec{t}, \vec{n}$ and $\vec{b}$ successively we obtain the following

$$C_p \frac{dT_t}{ds} = T \frac{dS}{ds} \tag{25}$$

$$V_t\, w_b\, \zeta = C_p \frac{dT_t}{dn} - T \frac{dS}{ds} \tag{26}$$

$$V_t\, w_n\, \zeta = T \frac{dS}{db} - C_p \frac{dT_t}{db} \tag{27}$$

From equation (25) we observe that the specific entropy remains uniform along a vortexline if the stagnation enthalpy is uniform along the same direction.

Eliminating ζ from (26) and (27) we obtain

$$T = \frac{C_p \left(w_n \dfrac{dT_t}{dn} + w_b \dfrac{dT_t}{db} \right)}{\left(w_b \dfrac{dS}{db} + w_n \dfrac{dS}{dn} \right)} \tag{28}$$

which gives the temperature.

Using (4) in (2) and eliminating p_t we obtain the following

$$\nabla \log p = -\frac{2\gamma w}{(\gamma - 1)\,(1 - w^2)} \nabla w + \frac{2\gamma}{\gamma - 1} \left(\frac{\vec{w} \wedge \text{curl}\, \vec{w} - q\, \vec{w}}{1 - w^2} \right) \tag{29}$$

Forming the scalar product by $\vec{t}, \vec{n}, \vec{b}$ we have

$$\frac{1}{p} \frac{dp}{ds} = -\frac{2\gamma w}{(\gamma - 1)\,(1 - w^2)} \frac{dw}{ds} - \frac{2\gamma q\, w_t}{(\gamma - 1)\,(1 - w^2)} \tag{30}$$

$$\frac{1}{p}\frac{dp}{dn} = -\frac{2\gamma w}{(\gamma-1)(1-w^2)}\frac{dw}{dn} + \frac{2\gamma\ (\zeta w_b - q w_n)}{(\gamma-1)(1-w^2)} \tag{31}$$

$$\frac{1}{p}\frac{dp}{db} = -\frac{2\gamma w}{(\gamma-1)(1-w^2)}\frac{dw}{db} - \frac{2\gamma\ (\zeta\ w_n + q w_b)}{(\gamma-1)(1-w^2)} \tag{32}$$

These give the variation of pressure along a vortexline, principal normal and binormal. The pressure remains uniform along a binormal to the vortexline if the vortexlines coincide with the streamlines, in this case additive heat has no effect along the normal component of acceleration field.

Operating curl (2) we obtain the following conditions

$$\frac{(\sigma' - \sigma'')\ q w_t}{1 - w^2} + \frac{d}{db}\left(\frac{-q w_n + \zeta w_b}{1 - w^2}\right) + \frac{d}{dn}\left(\frac{\zeta w_n + q w_b}{1 - w^2}\right) = 0 \tag{33}$$

$$\frac{d}{db}\left(\frac{q w_t}{1 - w^2}\right) + \frac{(\sigma'' + \tau)\ (\zeta w_b - q w_n)}{1 - w^2} + \frac{k''\ (\zeta w_n + q w_b)}{1 - w^2} - \frac{d}{ds}\left(\frac{\zeta w_n + q w_b}{1 - w^2}\right) = 0 \tag{34}$$

$$\frac{d}{dn}\left(\frac{q w_t}{1 - w^2}\right) - \frac{k q w_t}{1 - w^2} + \frac{d}{ds}\left(\frac{w_b \zeta - q w_n}{1 - w^2}\right) - \frac{k'\ (\zeta w_b - q w_n)}{1 - w^2} - \frac{(\sigma' - \tau)\ (\zeta w_n + q w_b)}{1 - w^2} = 0 \tag{35}$$

These constitute the basic compatibility conditions to be satisfied by a flow in vortexline geometry. The adiabatic phenomenon can be discussed as a special case of this investigation.

RESULTS OBTAINED

(1) Geometric conditions to be satisfied by triply orthogonal spatial curves of congruences are obtained, assigning one of the curve as a vortexline. The magnitude vorticity together with two basic results are obtained.

(2) Vorticity remains uniform along a vortexline, if the normal congruences are minimal.

(3) Conservation of mass is obtained in an elegant form.

(4) Total pressure remains uniform along vortexline if the flow is either complex-lamellar or fluid adiabatic. Analytic èxpression for temperature is obtained.

(5) Momentum equations are transformed into intrinsic form and the pressure remains uniform for the flow in which the streamline and vortexline coincide.

(6) Compatibility conditions governing the flow are obtained in intrinsic form, from which adiabatic case can be deduced as a special case.

REFERENCES

1. HICKS, B. I., *Qly. Appl. Maths.*, 6 (1948), 221.
2. PURUSHOTHAM, G. & MADHUSUDAN, *ZAMM*, 49 (1969), 365.
3. SURYANARAYAN, E. R., *ZAMM*, 48 (1968), 377.
4. KANWAL, R. P., *J. Math.*, 6 (1957), 621.
5. PURUSHOTHAM, G., *Appl. Sci. Res.*, 15 (1965), 23.