LAMINAR FLOW OF A VISCO-ELASTIC FLUID THROUGH RECTANGULAR CHANNEL

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Laminar flow of a visco-elastic fluid, characterised by three coefficients, through rectangular channel under the influence of exponential pressure gradient has been investigated. Two interesting cases have been discussed and analytical expressions have been obtained for the fluid velocity.

Liquids such as thick oils, pastes, paints, colloidal solutions and polymer solutions like polybutylene solutions are highly viscous. Their behaviour cannot be described by the classical hydrodynamic stress-strain velocity relations. Generalising the stress-strain velocity relations of classical hydrodynamics, the rheological behaviour of the non-Newtonian liquids has been studied by Rivlin[1] and Riener[2]. This generalisation affected the linearity of the usual stress-strain velocity relations. The attention of the authors[3-6], in the field of fluid-dynamics, has been diverted towards the non-linear character of stress-strain velocity relations of non-Newtonian fluids. Visco-elastic fluids are particular class of non-Newtonian fluids which exhibit appreciable elastic behaviour and stress-strain velocity relations are time dependent. Many authors[7-13] have studied the behaviour of different visco-elastic fluids. Longilois & Rivlin[11] have studied the steady state flow of visco-elastic fluids through non-circular tubes. Rivlin[6] has discussed some exact solutions of visco-elastic fluid flows. Jones & Walters[9-10] have investigated the oscillatory motion of visco-elastic liquid, characterised by three coefficients, between two coaxial circular cylinders and concentric spheres. In view of such an interest in the subject, in the present paper, the laminar flow of a visco-elastic fluid specified by three coefficients, one coefficient of viscosity and two relaxation time constants, through rectangular channel under the influence of exponential pressure gradient is investigated. Two interesting cases have been discussed and analytical expressions for fluid velocity have been obtained.

EQUATIONS OF MOTION

The equations of motion together with stress-strain velocity relations of visco-elastic fluid[9,10] are given by

$$\tau^{ij} = -pg^{ji} + \tau'^{ji} \tag{1}$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \tau'^{ij} = 2\mu \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) e^{ji},_j \tag{2}$$

$$e_{ij} = \frac{1}{2}\left(v^i,_j + v_{j,i}\right) \tag{3}$$

$$\rho\left(\frac{\partial v^i}{\partial t} + v^i,_j v_j\right) = \tau^{ij},_j \tag{4}$$

$$v^i,_i = 0 \tag{5}$$

where τ^{ij} and τ'^{ij} denote the stress and deviatoric stress tensors v^i ($i=1, 2, 3$) the components of velocity, g^{ij} are contravariant components of metric tensor, e_{ij} the strain rate of deformation tensor, λ_1 and λ_2 are relaxation time constants ($\lambda_1 \geqslant \lambda_2 \geqslant 0$), μ coefficient of viscosity, p the pressure and ρ the fluid density.

Operating $\left(1 + \lambda_1 \frac{\partial}{\partial t}\right)$ on the equation of motion (4) and using equation (1) and (2) one obtains

$$\rho \left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \left(\frac{\partial v^i}{\partial t} + v^i{}_{,j}\, v_j\right) = -\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) p_{,j}\, g^{ij} + 2\mu \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) e^{ij}{}_{,j} \tag{6}$$

FORMULATION OF THE PROBLEM

We shall investigate the flow of a visco-elastic fluid, described above, through a rectangular channel whose cross section is $(x^2-a^2)(y^2-b^2) = 0$, under the influence of exponential pressure gradient. Choosing axis of the channel along z-axis, the components of velocity are given by

$$v_1 = 0 \qquad v_2 = 0 \qquad v_3 = W(x, y, t) \tag{7}$$

Using the above relations, the equation of motion (6) can be expressed as

$$0 = -\frac{1}{\rho}\left(1 + \lambda_1 \frac{\partial}{\partial t}\right)\frac{\partial p}{\partial x} \tag{8}$$

$$0 = -\frac{1}{\rho}\left(1 + \lambda_1 \frac{\partial}{\partial t}\right)\frac{\partial p}{\partial y} \tag{9}$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right)\frac{\partial W}{\partial t} = -\frac{1}{\rho}\left(1 + \lambda_1 \frac{\partial}{\partial t}\right)\frac{\partial p}{\partial z} + \nu \left(1 + \lambda_2 \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right) \tag{10}$$

where ν is the kinematic coefficient of viscosity.

From equations (8) to (10), it follows that $\frac{1}{\rho}\left(\frac{\partial p}{\partial z}\right)$ is a function of t only. Since we have assumed the pressure gradient is exponential, we can write

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = \alpha\, e^{-m^2 t} \tag{11}$$

where α and m are real constants.

In view of the equation (11) we can express

$$W(x, y, t) = f(x, y)\, e^{-m^2 t} \tag{12}$$

Using relations (11) and (12) in equation (10) one obtains the following equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \Omega^2 (f + \alpha/m^2) = 0 \tag{13}$$

where $$\Omega^2 = \frac{m^2}{\nu} \cdot \frac{(1 - \lambda_1 m^2)}{(1 - \lambda_2 m^2)}$$

The problem is solved if we obtain the solution of the differential equation, subject to the following boundary conditions:

$$f(\pm a, y) = 0 \tag{14}$$

$$f(x, \pm b) = 0 \tag{15}$$

Since there is no slip of the fluid particles at the walls of the channel, the boundary conditions (15) can be satisfied by taking

$$f(x, y) = \sum_{0}^{\infty} F(x) \cos \left(\frac{2n+1}{2b}\right) \pi y \tag{16}$$

The remaining boundary conditions (14), then becomes

$$F(\pm a) = 0 \tag{17}$$

Using the relation (16) in (13) and taking

$$\alpha/m^2 = \frac{4\alpha}{m^2} \sum_{0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos \left(\frac{2n+1}{2b}\right) \pi y, \text{ we obtain}$$

$$\frac{d^2 F}{dx^2} - p_1^2 F = - \frac{4\Omega^2 \alpha (-1)^n}{m^2 \pi (2n+1)} \tag{18}$$

where $$p_1^2 = \frac{(2n+1)^2}{4b^2} \pi^2 - \Omega^2$$

The solution of the differential equation (18) is given by

$$F(x) = A \cosh p_1 x + B \sinh p_1 x + \frac{4\Omega^2 \alpha (-1)^n}{m^2 \pi p_1^2 (2n+1)} \tag{19}$$

Using the boundary conditions (17) in (19), we find that

$$\left.\begin{aligned} A &= -\frac{4\alpha\,\Omega^2\,(-1)^n}{m^2\,\pi\,p_1^2\,(2n+1)\cosh p_1 a} \\ B &= 0 \end{aligned}\right\} \tag{20}$$

From equations (19), (20) and (16), we can express

$$f(x,y) = \frac{4\Omega^2\,\alpha}{m^2\pi}\sum_0^\infty \frac{(-1)^n}{(2n+1)}\,\frac{1}{p_1^2}\left(1-\frac{\cosh p_1 x}{\cosh p_1 a}\right)\cos\frac{(2n+1)\,\pi y}{2b} \tag{21}$$

We shall now discuss two interesting cases of very small and large values of Ω :

Case (1)

When $|\Omega|$ is very small we can write $p_1 = \frac{(2n+1)\pi}{2b}$ approximately.

Therefore, the expression (21) simplifies to

$$f(x,y) = \frac{16b^2\,\alpha\,\Omega^2}{m^2\,\pi^3}\sum_0^\infty \frac{(-1)^n}{(2n+1)^3}\left(1-\frac{\cosh\frac{2n+1}{2b}\pi x}{\cosh\frac{2n+1}{2b}\pi a}\right)\cos\frac{(2n+1)}{2b}\pi y \tag{22}$$

From relation (12), it follows that

$$W(x,y,t) = \frac{16b^2\,\alpha\,\Omega^2\,e^{-m^2 t}}{m^2\,\pi^3}\sum_0^\infty \frac{(-1)^n}{(2n+1)^3}\left(1-\frac{\cosh\frac{2n+1}{2b}\pi x}{\cosh\frac{2n+1}{2b}\pi a}\right)\cos\frac{(2n+1)}{2b}\pi y \tag{23}$$

which is the velocity of the fluid in the present case.

Case (2)

When $|\Omega|$ is very large writing $p_1^2 = -p_1'^2$, the solution of the differential equation (18) can be expressed as

$$F(x) = A'\cos p_1' x + B'\sin p_1' x + \frac{4\Omega^2\,\alpha\,(-1)^{n+1}}{m^2\pi\,p_1'^2\,(2n+1)} \tag{24}$$

where A' and B' are constants to be determined subject to the boundary conditions (17).

Using the boundary conditions (17), one obtains

$$\left.\begin{aligned} A' &= \frac{4\Omega^2 \alpha (-1)^n}{m^2 \pi p_1'^2 (2n+1) \cos p_1' a} \\ B' &= 0 \end{aligned}\right\} \tag{25}$$

On using equation (25) in (24) we obtain

$$F(x) = \frac{4\Omega^2 \alpha}{m^2 \pi} \sum_0^\infty \frac{(-1)^n}{(2n+1)} \frac{1}{p_1'^2} \left(\frac{\cos p_1' x}{\cos p_1' a} - 1 \right) \tag{26}$$

From equations (16) and (26), it follows that

$$f(x,y) = \frac{4\Omega^2 \alpha}{m^2 \pi} \sum_0^\infty \frac{(-1)^n}{(2n+1)} \frac{1}{p_1'^2} \left(\frac{\cos p_1' x}{\cos p_1' a} - 1 \right) \cos \frac{(2n+1)}{2b} \pi y \tag{27}$$

For very large values of $|\Omega|$, we can write $p_1' \simeq \Omega$. Therefore the expression (27) simplifies to

$$f(x,y) = \frac{4\alpha}{m^2 \pi} \sum_0^\infty \frac{(-1)^n}{(2n+1)} \left(\frac{\cos \Omega x}{\cos \Omega a} - 1 \right) \cos \frac{(2n+1)}{2b} \pi y \tag{28}$$

Using relation (28) in (12), we get

$$W(x, y, t) = \frac{4\alpha e^{-m^2 t}}{m^2 \pi} \sum_0^\infty \frac{(-1)^n}{(2n+1)} \left(\frac{\cos \Omega x}{\cos \Omega a} - 1 \right) \cos \frac{(2n+1)}{2b} \pi y \tag{29}$$

which determines the fluid velocity in the present case.

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