# HYDROMAGNETIC FLOW BETWEEN TWO ROTATING COAXIAL DISCS

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**This paper relates to the steady flow of an electrically conducting incompressible viscous fluid between two parallel coaxial rotating discs with a transverse magnetic field when the discs are rotating in the same direction with the same velocity and there is a source at the centre.**

In the present article the hydromagnetic source flow of a viscous, incompressible and electrically conducting fluid between two parallel coaxial rotating discs has been analysed An analysis of the velocity distribution has been made when there is a constant magnetic field of strength $B_0$ in a direction perpendicular to the discs. The analysis is limited to the case of small magnetic Reynolds number. Similar problems of source flow for the non-magnetic case have been recently studied by Kreith & Peube[1,2], Khan[3], Breiter & Pohlhausen[4] and Geiger, Fara & Street[5] when the discs are rotating with the same velocities and by Kreith & Viviand[6] when the two discs are rotating with different velocities. Such type of analysis may find applications in design of viscosity pumps[7,8], rotating heat exchangers[9] and air thrust bearing[10].

## FUNDAMENTAL EQUATIONS AND BOUNDARY CONDITIONS

Let us take the axis of rotation of the two discs as $\bar{z}$-axis and let the two plates be situated at $\bar{z} = \pm a$. Consider the flow of an incompressible fluid between two parallel rotating discs with a source at the centre.

The governing hydromagnetic equations in cylindrical polar coordinates $(\bar{r}, \bar{\theta}, \bar{z})$ are

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{r}} + \bar{w}\frac{\partial \bar{u}}{\partial \bar{z}} - \frac{\bar{v}^2}{\bar{r}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{r}} + \nu\left[\frac{\partial}{\partial \bar{r}}\left(\frac{\partial \bar{u}}{\partial \bar{r}} + \frac{\bar{u}}{\bar{r}}\right) + \frac{\partial^2 \bar{u}}{\partial \bar{z}^2}\right] - \frac{\sigma B_o^2}{\rho}\bar{u} \tag{1}$$

$$\bar{u}\frac{\partial \bar{v}}{\partial \bar{r}} + \bar{w}\frac{\partial \bar{v}}{\partial \bar{z}} + \frac{\bar{u}\cdot\bar{v}}{\bar{r}} = \nu\left[\frac{\partial}{\partial \bar{r}}\left(\frac{\partial \bar{v}}{\partial \bar{r}} + \frac{\bar{v}}{\bar{r}}\right) + \frac{\partial^2 \bar{v}}{\partial \bar{z}^2}\right] - \frac{\sigma B_o^2}{\rho}\bar{v} \tag{2}$$

$$\bar{u}\frac{\partial \bar{w}}{\partial \bar{r}} + \bar{w}\frac{\partial \bar{w}}{\partial \bar{z}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{z}} + \nu\left[\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial \bar{w}}{\partial \bar{r}}\right) + \frac{\partial^2 \bar{w}}{\partial \bar{z}^2}\right] \tag{3}$$

and the equation of continuity is

$$\frac{\partial}{\partial \bar{r}}\left(\bar{r}\cdot\bar{u}\right) + \frac{\partial}{\partial \bar{z}}\left(\bar{r}\cdot\bar{\omega}\right) = 0 \tag{4}$$

where $u, v, w$ denote, respectively, the radial, transverse and axial components of velocity, $\bar{p}$ denotes the fluid pressure, $B_0$ the strength of the uniform axial magnetic field, $\sigma$ the electrical conductivity, $\nu$ the kinematic viscosity of the fluid and $\rho$ its density. From symmetry of flow all quantities are independent of $\bar{\theta}$.

If the two discs rotate with angular velocity $\omega$ and the strength of source is $Q$, the boundary conditions are

$$\left.\begin{aligned} \bar{u} &= 0 \\ \bar{v} &= \bar{r}\,\omega \\ \bar{w} &= 0 \end{aligned}\right\} \quad \text{at } \bar{z} = \pm a \tag{5}$$

$$\int_{-a}^{+a} 2\pi \bar{r}\,\bar{u}\, d\bar{z} = Q \tag{6}$$

Appropriate dimensionless variables are defined by the following relations :

$$r = \bar{r}/\sqrt{\nu/\omega}, \quad z = \bar{z}/\sqrt{\nu/\omega}, \quad u = \bar{u}/\sqrt{\nu\omega}, \quad v = \bar{v}/\sqrt{\nu\omega}, \quad w = \bar{w}/\sqrt{\nu\omega}, \quad p = \bar{p}/\rho\nu\omega$$

In terms of these dimensionless variables, equations (1) to (4) become

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{\partial p}{\partial r} + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2} - Mu \tag{7}$$

$$u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{u \cdot v}{r} = \frac{\partial}{\partial r}\left(\frac{\partial v}{\partial r} + \frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2} - Mv \tag{8}$$

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial z^2} \tag{9}$$

$$\frac{\partial}{\partial r}(r \cdot u) + \frac{\partial}{\partial z}(r \cdot w) = 0 \tag{10}$$

and the boundary conditions become

$$\left.\begin{aligned} u &= o \\ v &= r \\ w &= o \end{aligned}\right\} \quad \text{at} \quad z = \pm d \tag{11}$$

$$r \cdot \int_{-d}^{+d} u\, dz = K \tag{12}$$

where

$$K = Q\omega^{\frac{1}{2}}/2\pi\nu^{3/2}, \qquad d = a/\sqrt{\nu/\omega} \qquad \text{and} \quad M = \sigma B_o^2/\rho\,\omega.$$

## SOLUTION OF THE PROBLEM

Applying boundary layer approximations, the above system of equations reduces

$$u\frac{\partial u}{\partial r} - \frac{v^2}{r} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial r} + \frac{\partial^2 u}{\partial z^2} - Mu \tag{13}$$

$$u \frac{\partial v}{\partial r} + \frac{u \cdot v}{r} + w \frac{\partial v}{\partial z} = \frac{\partial^2 v}{\partial z^2} - Mv \tag{14}$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \tag{15}$$

It is convenient to introduce the non-dimensional tangential velocity $V$ relative to the discs instead of $v$ which is the non-dimensional tangential velocity in the fixed system of coordinates.

$$V = v - r \tag{16}$$

One then obtains

$$u \frac{\partial u}{\partial r} - \frac{V^2 + 2Vr + r^2}{r} + w \frac{\partial u}{\partial z} = - \frac{\partial p}{\partial r} + \frac{\partial^2 u}{\partial z^2} - Mu \tag{17}$$

$$u \left( \frac{\partial V}{\partial r} + 1 \right) + \frac{u (V + r)}{r} + w \frac{\partial V}{\partial z} = \frac{\partial^2 V}{\partial z^2} - M (V + r) \tag{18}$$

and the boundary conditions (11) then become

$$\left. \begin{aligned} u (r, \pm d) &= 0 \\ V (r, \pm d) &= 0 \\ w (r, \pm d) &= 0 \end{aligned} \right\} \tag{19}$$

Assuming that the velocity of the fluid relative to the discs is small so that the quadratic terms are negligible (this condition is well satisfied for large radii), we obtain from equations (17) and (18)

$$\frac{\partial^2 u}{\partial z^2} + 2V - Mu = \frac{\partial p}{\partial r} - r \tag{20}$$

$$u = \frac{1}{2} \cdot \frac{\partial^2 V}{\partial z^2} - \frac{1}{2} M (V + r) \tag{21}$$

In these equations $\partial p / \partial r$ does not depend upon $z$ because of the boundary layer assumption that the pressure does not depend on $z$. The unknown $w$ does not appear and so it is possible to solve these equations separately; $w$ can be determined from the equation continuity.

Eliminating $u$ from equation (20) with the help of equation (21), we get

$$\frac{d^4 V}{dz^4} - 2M \frac{d^2 V}{dz^2} + (M^2 + 4) V = F(r) \tag{22}$$

where

$$F(r) = \left[ 2 \left( \frac{\partial p}{\partial r} - r \right) - M^2 r \right] \tag{23}$$

Solving equation (22) and applying the boundary conditions, we get

$$V = A_1 \cosh (\alpha z) \cos (\beta z) + B_1 \sinh (\alpha z) \sin (\beta z) + F_1 (r) \tag{24}$$

$$u = A_2 \cosh (\alpha z) \cos (\beta z) + B_2 \sinh (\alpha z) \sin (\beta z) + F_2 (r) \tag{25}$$

where

$$\alpha = \gamma^{1/2} \cos \theta/2, \qquad \beta = \gamma^{1/2} \sin \theta/2$$

$$\gamma = (M^2 + 4)^{1/2}, \quad \theta = \tan^{-1} 2/M$$

$$F_1(r) = F(r)/\gamma^2, \qquad F_2(r) = -\frac{1}{2} M \left\{ F_1(r) + r \right\}$$

$$A_1 = (F_2 X_2 - F_1 X_4) / (X_1 X_4 - X_2 X_3)$$

$$B_1 = (F_1 X_3 - F_2 X_1) / (X_1 X_4 - X_2 X_3)$$

$$A_2 = \frac{1}{2} \left\{ (\alpha^2 - \beta^2) - M \right\} A_1 + \alpha \beta B_1$$

$$B_2 = \frac{1}{2} \left\{ (\alpha^2 + \beta^2) - M \right\} B_1 - \alpha \beta A_1$$

and

$$X_1 = \cosh(\alpha d) \cos(\beta d)$$

$$X_2 = \sinh(\alpha d) \sin(\beta d)$$

$$X_3 = \frac{1}{2} \left\{ (\alpha^2 - \beta^2) - M \right\} X_1 - \alpha \beta X_2$$

$$X_4 = \frac{1}{2} \left\{ (\alpha^2 + \beta^2) - M \right\} X_2 + \alpha \beta X_1$$

The factor $F(r)$ is determined by the condition that the mass flow between the discs is constant for every cross-section $r$=constant. Let the strength of the source (measured by volume) be $Q$. The flow through a surface $r$=constant extending between the discs is given by equation (12), i.e.

$$r. \int_{-d}^{+d} u \, dz = K.$$

Substituting the value of $u$ and integrating, we get

$$K = 2r (A_2 K_1 + B_2 K_2 + F_2 d) \tag{26}$$

where

$$K_1 = \left\{ \alpha \sinh(\alpha d) \cos(\beta d) + \beta \cosh(\alpha d) \sin(\beta d) \right\} / (\alpha^2 + \beta^2)$$

$$K_2 = \left\{ \alpha \cosh(\alpha d) \sin(\beta d) - \beta \sinh(\alpha d) \cos(\beta d) \right\} / (\alpha^2 + \beta^2)$$

Substituting the values of $A_2$ and $B_2$ in equation (26) and simplifying, we get

$$K = 2r \left[ F(r) \left\{ (K_1 K_3 + K_2 K_5) \frac{1}{\gamma^2} - (K_1 K_4 + K_2 K_6 + d) \frac{M}{2\gamma^2} \right\} - \right.$$
$$\left. - (K_1 K_4 + K_2 K_6 + d) \frac{M}{2} r \right] \tag{27}$$

where

$$K_3 = \left[\alpha\,\beta\,X_3 - \frac{1}{2}\left\{(\alpha^2 - \beta^2) - M\right\} X_4\right] / (X_1 X_4 - X_2 X_3)$$

$$K_4 = \left[\frac{1}{2}\left\{(\alpha^2 - \beta^2) - M\right\} X_2 - \alpha\,\beta\,X_1\right] / (X_1 X_4 - X_2 X_3)$$

$$K_5 = \left[\frac{1}{2}\left\{(\alpha^2 + \beta^2) - M\right\} X_3 - \alpha\,\beta\,X_4\right] / (X_1 X_4 - X_2 X_3)$$

$$K_6 = \left[\alpha\,\beta\,X_2 - \frac{1}{2}\left\{(\alpha^2 + \beta^2) - M\right\} X_1\right] / (X_1 X_4 - X_2 X_3).$$

From equation (27) we find the function $F(r)$

$$F(r) = K / 2r\,C_1 + C_2\,r / C_1 \tag{28}$$

where

$$C_1 = \frac{1}{\gamma^2}\left(K_1 K_3 + K_2 K_5\right) - \frac{M}{2\gamma^2}\left(K_1 K_4 + K_2 K_6 + d\right)$$

$$C_2 = \frac{1}{2}\,M\left(K_1 K_4 + K_2\,K_6 + d\right)$$

For the non-magnetic case, i.e. when $M = 0$, equations (24) and (25) reduce to

$$V = \frac{q}{r}\left[A_3 \sinh z \sin z + B_3 \cosh z \cos z - 1\right] \tag{29}$$

$$u = \frac{q}{r}\left[A_3 \cosh z \cos z - B_3 \sinh z \sin z\right] \tag{30}$$

where

$$A_3 = 2\,\sinh\,d\,\sin\,d \Big/ \left\{\cosh\,2d + \cos 2d\right\}$$

$$B_3 = 2\,\cosh\,d\,\cos\,d \Big/ \left\{\cosh\,2d + \cos\,2d\right\}$$

$$q = K\left\{\cosh\,2d + \cos\,2d\right\} \Big/ \left\{\sinh\,2d - \sin 2d\right\}$$

These results agree with those obtained by Breiter & Pohlhausen[4] if the latter results are made dimensionless using our substitutions.

## DISCUSSION

Curves have been drawn showing the variation of non-dimensional radial velocity $u$ and the non-dimensional transverse velocity $V$ relative to the discs with $z$. The constants appearing in the values of $u$ and $V$ have been assumed to be

$$M = 0{\cdot}5, \qquad d = 1, \qquad K = 3, \qquad r = 10$$

From Fig. 1 it is seen that the value of $u$ is zero at the discs. It increases rapidly as the non-dimensional distance from the plate increases from 0 to 0·2. It attains its greatest value at $z \simeq \pm 0{\cdot}8$. The value then decreases rapidly and attains almost zero value in a region midway between the two discs bounded by the planes $z \simeq 0{\cdot}1$.

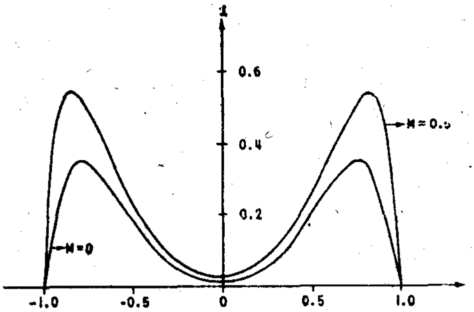


Fig. 2—Variation of $u$ with $z$.

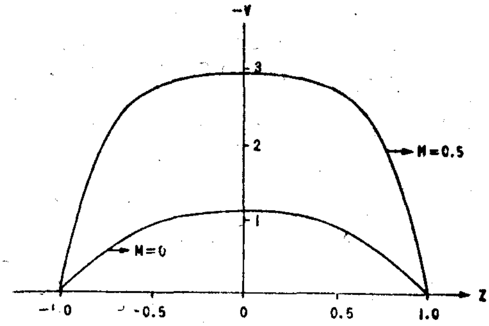


Fig. 2—Variation of $v$ with $z$.

The pattern of variation of $u$ with $z$ for the magnetic and non-magnetic cases is almost the same. The effect of magnetic field is to increase the velocity. This effect is greatest at $z \doteq \pm 0 \cdot 8$.

From Fig. 2 we see that the value of $V$ is always negative. It is zero at the discs and increases numerically as the non-dimensional distance from the discs increases from 0 to $0 \cdot 4$. The increase is then gradual till it is almost constant in the region bounded by the planes $z \doteq \pm 0 \cdot 2$.

The effect of magnetic field is again to increase the velocity and the pattern of variation of $V$ with $z$ for the magnetic case is the same as for the non-megnetic case.

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