APPLICATION OF GROUP THEORY TO THE SIMILARITY SOLUTIONS OF THE POWER LAW FLUIDS FLOW PAST A SUDDENLY ACCELERATING PLATE

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(*Received* 3 *March* 1969)

The motion of a power law fluid past a suddenly accelerated wall which moves with a constant velocity U (t) parallel to the wall has been considered for the similarity analysis. Recently this problem has been discussed and two possible group of transformation have been used by T.Y. Na[1]. In this paper the third possible transformation has been found. The variation of the wall velocity U (t) with time, t, has been worked out.

NOTATIONS

t = time

μ = coefficient of viscosity

v = coefficient of kinematic viscosity

$U(t)$ = velocity of the plate

$<_{ij}$ = components of the deviatoric stress tensor

e_{ij} = strain-rate components

k,n = consistency and flow behaviour indices

$a, a_1, a_2, a_3, \beta, \beta_1 \beta_2, \beta_3$, A, B, b, m, a_i, β_i = certain constants

ξ,η = independent similarity variables

p,q = certain constants in terms of β_i

m_1,m_2 = constants in terms of β_i

F_m = function of η

F_p = function of y

G_q = function of ξ

The power law fluids have been found of great interest in Chemical Engineering and Technology and their flow properties have been discussed in detail[1-5]. Such fluids satisfy the rheological equation[6-7].

$$<_{ij} = k \left| \sum_{l=1}^{3} \sum_{m=1}^{3} e_{lm}\, e_{ml} \right|^{\frac{n-1}{2}} e_{ij} \tag{1}$$

where

$$e_{ij} = \frac{\partial u_i}{x_j} + \frac{\partial u_j}{\partial x_i} \tag{2}$$

are the strain-rate components, (u_1, u_2, u_3) are the velocity components, $<_{ij}$ are the components of the deviatoric stress tensor, k and n are usually called the consistency and flow-behavior indices respectively.

In this note, we consider the flow of such fluids past a suddenly accelerated wall which moves with a constant velocity $U(t)$ parallel to the wall. We select x-axis along the wall and y-axis perpendicular to the wall drawn into the fluid. It is the purpose of this note to find the invatiants and the forms of $U(t)$ for which similarity solutions exist. Let $[u(y, t), 0]$ denote the components of velocity at time t at a distance y from the plane. (Fig. 1) Thus the boundary layer equations are reduced to

$$\rho \frac{\partial u}{\partial t} = - \frac{\partial p}{\partial x} + \mu \frac{\partial}{\partial y} \left[\left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y} \right] \tag{3}$$

$$0 = - \frac{\partial p}{\partial y} \tag{4}$$

The equation of continuity is satisfied from the forms of u and v, we have selected From (3) and (4), $\frac{\partial p}{\partial x}$ is a function of time or constant. We consider the case when there is a pressure gradient and so the simplified equation can be written[5] as

$$\frac{\partial u}{\partial t} = \nu \frac{\partial}{\partial y} \left(\left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y} \right) \tag{5}$$

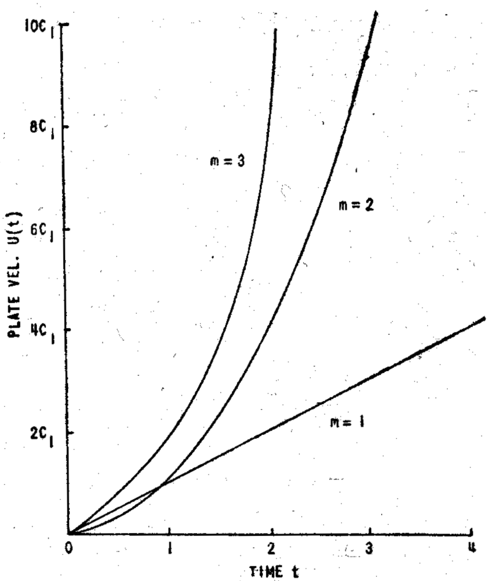

Fig. 1—Variation of wall velocity with time.

with the boundary conditions

$$\left.\begin{array}{llll} y=0 : & u=U(t) & \text{for} & t \geqslant 0 \\ y=\infty : & u=0 & \text{for} & \text{all values of } t \\ y \geqslant 0 : & u=0 & \text{for} & t<0 \end{array}\right\} \tag{6}$$

Na[1] has considered a semi-infinite body of non-Newtonian fluid, bounded on one side by a flat plate, which is initially at rest. At time $t=0$, the plate is set in motion parallel to itself with velocity $U(t)$. The desirable forms of $U(t)$ have been found in reference[7] for which the similar solutions are possible. Two group transformations (*i*) $t=A^{\alpha_1}\bar{t}$, $y=A^{\alpha_2}\bar{y}$ $u=A^{\alpha_3}\bar{u}$ and (*ii*) $t=\bar{t}+\beta_1 b$, $y=e^{\beta_2 b}\bar{y}$, $u=e^{\beta_3 b}\bar{u}$ are selected where α_i, $i=1, 2, 3$; β_j, $j=1, 2, 3$; A and b are certain constants. It has been shown that for similar solutions, the wall velocity must vary as (*i*) $U(t)=c_1 t^m$, (*ii*) $U(t)=c_2\, e^{[(n+1)/(n-1)]qt}$ respectively where c_1, c_2, m, n and q are constants.

Two possible cases have been found by Yen Na[1] and third has been found in this note. It has been shown here that case I of Na[1] is a particular case of the present note. Case II of Na[1] is also obtainable from the third case.

SOLUTION OF THE PROBLEM

Case (*i*)

One parameter group of transformation is chosen in the following form

$$t=A^{\alpha_1}\bar{t}+B(A^{\alpha_1}-1), \; y=A^{\alpha_2}\bar{y}, \; u=A^{\alpha_3}\bar{u} \tag{7}$$

where α_1, α_2, α_3, A and B are certain constants. Substituting from (7) into (5), equating the powers of A on both sides and defining m and b by

$$m=\frac{\alpha_3}{\alpha_1}, \; b=\frac{\alpha_2}{\alpha_1} \tag{8}$$

we get the following relations between b and m

$$b=\frac{1+(n-1)m}{n+1} \tag{9}$$

From (7), we easily find that

$$\frac{y}{(t+B)^b}=\frac{\bar{y}}{(\bar{t}+B)^b} \tag{10}$$

$$\frac{u}{(t+B)^m}=\frac{\bar{u}}{(\bar{t}+B)^m} \tag{11}$$

Thus the absolute invariants are

$$\eta=\frac{y}{(t+B)^{[1+m(n-1)]/(n+1)}} \tag{12}$$

$$F_m(\eta)=\frac{u}{(t+B)^m} \tag{13}$$

where m is an arbitrary constant. Substituting (12) and (13) into (5), we get

$$\nu \frac{d}{d\eta}\left(\left|\frac{dF_m}{d\eta}\right|^{n-1}\frac{dF_m}{d\eta}\right)+\frac{1+m(n-1)}{n+1}\,\eta\,\frac{dF_m}{d\eta}-m\,F_m=0 \qquad (14)$$

which is equation 7 of Na[1].

Thus for the similarity solutions, the wall velocity is

$$U(t)=c_1\,(t+B)^m \qquad (15)$$

since the reduced boundary conditions are

$$\left.\begin{aligned}&\eta=0\;;F_m=c_1 &&\text{for } t\geqslant 0\\&\eta=\infty\;;F_m=0 &&\text{for all values of } t\\&\eta\geqslant 0\;;F_m=0 &&\text{for } t<0\end{aligned}\right\} \qquad (16)$$

For $m=0$ and $n\neq 1$, this reduces to solutions of Bird[8] and Wells[9]. For $B=0$, we get case I of Na[1] and heating starts at time, $t=0$. Thus case I of Na[1] is a particular case of the present discussion. Replacing B by A, m by α, $Fm(\eta)$ by $F'(\eta)$, we get the case I of Nanda[10].

Case (ii)

The second group of transformations are

$$t=e^{\beta_1 b}\,\bar{t}\,,\; y=e^{\beta_2 b}\,\bar{y},\; u=e^{\beta_3 b}\,\bar{u} \qquad (17)$$

substituting (17) into (5), we get the similar results as in case I of Na[1]. In fact the above transformations are particular forms of $t=A^{a_1}\,\bar{t}$ etc. of Na[1].

Case (iii)

The following transformations are selected

$$t=e^{\beta b}\,\bar{t}+\beta_1 b\,,\; y=e^{\beta_2 b}\,\bar{y}\,,\; u=e^{\beta_3 b}\,\bar{u} \qquad (18)$$

substituting (18) into (5) and equating the powers of e^b on both sides, we get

$$\beta_3=n\,\beta_3-\beta_2\,(n+1)+\beta \qquad (19)$$

For the Newtonian fluid, $n=1$, we get $\beta_2=0$ if $\beta=0$. In this case the absolute invariants are found as in Na[1] *viz.*,

$$\frac{u}{e^{pt}}=\frac{\bar{u}}{e^{p\bar{t}}} \qquad (20)$$

where

$$p=\beta_3/\beta_1 \qquad (21)$$

$$F_p\,(y) = \frac{u}{e^{pt}} \tag{22}$$

Thus the equation (5) from (22) gives

$$\frac{\partial^2 F_p}{dy^2} - \frac{p}{\nu}\,F_p = 0 \tag{23}$$

which is equation 12 of Na[1]

If $n \neq 1$, we get from (19)

$$\beta_3\,(n-1) + \beta = (n+1)\,\beta_2$$

or,

$$\frac{\beta_3}{\beta}(n-1) + 1 = (n+1)\,\beta_2/\beta$$

or,

$$m_2 = \frac{m_1\,(n-1)+1}{n+1} = \beta_2/\beta \tag{24}$$

where $m_1 = \beta_3/\beta$

For $\beta = 0$, we get case II of Na[1] and the absolute invariants are

$$\xi = \frac{y}{e^{qt}} \tag{25}$$

$$Gq\,(\xi) = \frac{u}{e^{\{(n+1)/(n-1)\}qt}} \tag{26}$$

where

$$q = \beta_2/\beta_3 \tag{27}$$

and from (25), (26) and (5), we get

$$\nu\frac{d}{d\xi}\left(\left|\frac{d\,G_q}{d\xi}\right|^{n-1}\frac{d\,G_q}{d\xi}\right) + q\xi\,\frac{d\,G_q}{d\xi} - \frac{n+1}{n-1}\,q\,G_q = 0 \tag{28}$$

which is equation 18 of Na[1]

Replacing $G_q\,(\xi)$ by $F'\,(\eta)$, $\frac{q\,(n+1)}{(n-1)}$ by m we get case II of Nanda (reference 2, eqn. 21).

CONCLUSION

The transformation groups discussed by Bird *et al*[7] are the particular cases of case (*iii*), eqn. (18). For $\beta_1 = 0$, we get case I of reference 7 and case (*ii*) of the present note. For $\beta = 0$, we get case II of reference 1. For $\beta_1 \neq 0$ and $\beta \neq 0$, it is concluded after calculations that the absolute invariants cannot be found.

REFERENCES

1. NA, T. Y., *A.I.A.A. J.*, 3, No. 2 (1964), Tech. note, 378.
2. WILKINSON, W. L., "Non-Newtonian Fluids" (Pergamon Press), 1960.
3. METZNER, A. B., 'Advances in Chemical Engineering', Vol. 1 (Academic Press, New York) 1956.

4. METZNER, A. B., "Handbook of Fluid Dynamics (McGraw Hill Book Co., New York), 1961.
5. KAPUR, J. N., *J. Phys. Soc.*, Japan, 17 (1962), 1303.
6. KAPUR, J. N., *J. Phys. Soc.*, Japan, 18 (1963), 438.
7. BIRD, R. B., STEWART W. E. & LIGHTFOOT, E. N., Transport Phenomena (John Wiley and Sons, New York) 1960.
8. BIRD, R. B., *Amer, Inst. Chem. Engrs.*, 5 (1959), 565.
9. WELLS, C. S. Jr., *A I A A J.*, 2 (1964), 951.
10. NANDA, R. S., *J. Sci. Engg. Res.*, 7 (1963) 251.