RADIAL DEFORMATIONS IN A COMPRESSIBLE DIELECTRIC CYLINDRICAL BLOCK

K. S. SARMA

Regional Engineering College, Warangal

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In this paper, the problem of bending of an initially curved cylindrical block of compressible dielectric material is solved when a uniform surface charge is applied on the outer surface of the block. It is observed that the surface charge applied to the outer surface does not effect the stress distribution. The results for a material having a particular strain energy function have been discussed in detail. The results for a cylinder under uniform pressures are indicated.

Recently, Toupin[1] derived the fundamental equations for a compressible dielectric material. Eringen[2,3] reoriented the topic using the variational technique and solved the problems of pure shear of a block and uniform extension of a circular cylinder of incompressible dielectric material. Singh & Pipkin[4] discussed the possible deformations in an incompressible dielectric.

In this paper, we consider the problem of bending of an initially curved block of a compressible dielectric material with a uniform surface charge on the outer surface of the cylindrical block. It is observed that the stress distribution does not depend on the surface charge. The transverse stress t^{θ}_{θ} and axial stress t^{z}_{z} are both independent of the polarization $P(r)$ explicitly. The results for a material having particular strain energy function are discussed in detail.

ROTATION AND FORMULAE

The basic equations of homogeneous isotropic dielectric are, in Eringen's[3] notation as follows :

(*i*) *Field Equations*

$$t^{k}_{l;k} + \rho f_l = 0 \tag{1}$$

$$_{L}E^{k} - \phi_{,k} = 0 \tag{2}$$

$$\epsilon_0 \nabla^2 \phi - \text{div}\, \vec{P} + q_f = 0 \text{ in } V_d \tag{3}$$

where V_d is the volume that the dielectric occupies, semicolon and comma stand for covariant and ordinary partial differentiation respectively, t^{k}_{l} is the Cauchy stress tensor, f_l is the body force per unit volume, ρ the density, $_{L}E^{k}$ the local electric field, ϕ the electrostatic potential, $\vec{P}$ the polarization, ϵ_0 a constant and q_f is the volume free charge.

(ii) Boundary conditions

$$[\![t_l^k]\!] n_k = 0 \tag{4}$$

$$[\![\epsilon_0 \phi,^k - P^k]\!] n_k + \omega_f = 0 \text{ on } S_d \tag{5}$$

where S_d is the surface of the dielectric. The Cauchy stress tensor t_l^k is defined by

$$t_l^k = L_l^k - M_l^k \tag{6}$$

where L_l^k, the local stress tensor and ${}_L E^k$ are given by the constitutive relations (8) and (9) and M_l^k, the Maxwell stress tensor is given by

$$M_l^k = \epsilon_0 [\phi,^k \phi,_l - \tfrac{1}{2} \phi,^m \phi,_m] \delta_l^k \tag{7}$$

In (5), n_k denotes the exterior normal to S_d, ω_f denotes the surface free charge and double bracket stands for discontinuity across the surface.

(iii) Constitutive relations

The local electric field and the local stress tensor are given by :

$$L_l^k = \frac{2\rho}{\rho_0} \Big[I_3 \frac{\partial \Sigma}{\partial I_3} \delta_l^k + \left(\frac{\partial \Sigma}{\partial I_1} + I_1 \frac{\partial \Sigma}{\partial I_2} \right) c_l^{-1\,k} - \frac{\partial \Sigma}{\partial I_2} c_l^{-2k}$$
$$+ \frac{\partial \Sigma}{\partial I_4} c_m^{-1\,k} P^m P^l + \frac{\partial \Sigma}{\partial I_5} c_m^{-2\,k} P^m P^l + \frac{\partial \Sigma}{\partial I_5} c_m^{-1\,k} c_l^{-1n} P^m P_n \Big] \tag{8}$$

$${}_L E^k = \frac{2\rho}{\rho_0} \left[\frac{\partial \Sigma}{\partial I_4} c_m^{-1\,k} + \frac{\partial \Sigma}{\partial I_5} c_m^{-2\,k} + \frac{\partial \Sigma}{\partial I_6} \delta_m^k \right] P^m \tag{9}$$

where $\Sigma = \Sigma$ (I_r, P), (r = 1 to 6) and I_r are invariants based on the strain and polarization. The strain invariants are given by

$$I_1 = (c^{-1})_i{}^i, \; 2 I_2 = I_1^2 - (c^{-1})^{ij} (c^{-1})_{ij}, \; I_3 = | c^{-1\,i}{}_j |$$
$$I_4 = c_l^{-1k} P^l P_k, \; I_5 = c_l^{-2k} P^l P_k, \; I_6 = P^2 ; \; \frac{\rho}{\rho_0} = 1 / \sqrt{I_3} \tag{10}$$

and $| c_j^{-1\,i} |$ is the determinant of the matrix $\| c_j^{-1\,i} \|$.

BENDING OF AN INITIALLY CURVED CUBOID

Let X^α be the cylindrical coordinates (R, v, Z) to describe the cuboid before deformation given by $R = a_1$ and $R = b_1$ $(a_1 > b_1)$, $v = \pm v_0$ and $Z = \pm Z_0$. Let x^k be also the cylindrical coordinates (r, θ, z) with the same origin and z-axis and the cuboid after deformation be given by $r = c_1$ and $r = d_1$ $(c_1 > d_1)$ and $\theta = \pm \theta_0$ and $z = \pm z_0$.

Since the surface R = a constant goes into r = constant, planes $v = a$ constant to θ = a constant and the plane Z = a constant to z = a constant, we have

$$R = R(r) \quad v = A\theta, \quad \text{and } Z = \lambda z \tag{11}$$

where $A = v_0 / \theta_0$; $\lambda = Z_0 / z_0$. The deformation tensor c_l^{-1k} is given by

$$\| c_l^{-1k} \| = \begin{bmatrix} 1/R'^2 & 0 & 0 \\ 0 & r^2 / A^2 R^2 & 0 \\ 0 & 0 & 1/\lambda^2 \end{bmatrix} \tag{12}$$

where dashes denote differentiations with respect to r.

Electrostatic Potential, Maxwell and local fields :

Because the deformation is radial, let us assume the electrostatic potential ϕ, polarization vector $\vec{P}$ and electric field $\vec{E}$ to be functions of the radius r and given by

$$\phi = \phi(r), \quad \vec{P} = [P(r), 0, 0], \quad \vec{E} = [E(r), 0, 0]$$

The field equation (3) is given by

$$\epsilon_0 \frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} = \frac{1}{r}\frac{d(rP)}{dr} \tag{13}$$

in the absence of volume free charge in the body and space outside it. Equation (13) gives ϕ for the three regions $0 < r < d_1$; $d_1 < r < c_1$ and $r > c_1$ respectively as

$$\begin{aligned} \epsilon_0 \phi &= F \qquad r < d_1 \\ \epsilon_0 \phi &= B + C \log r + \int^r P(\xi)\, d\xi \, ; \; d_1 < r < c_1 \\ \epsilon_0 \phi &= D + G \log r \qquad r > d_1 \end{aligned} \tag{14}$$

On using the regularity condition of ϕ on the z axis, the constants F, B, C, D, G are to be determined from the boundary conditions viz., (5) on both the surfaces and there is a surface charge ω_f on the outer surface of the body. These give

$$C = 0, \quad G = -c_1 \omega_f$$

The other constants are left arbitrary as they do not enter the stress distribution. It is interesting to note that the electrostatic potential ϕ inside the body is not effected by the surface charge density on the outer surface of the body.

From $\vec{M^E} = -\text{grad}\,\phi$, we get the Maxwell electric field as

$$M^{E^r} = -\frac{P}{\epsilon_0}, \; M^{E^\theta} = M^{E^z} = 0 \tag{15}$$

and Maxwell stress field from (7) is given by

$$M^r_r = -M^\theta_\theta = -M^z_z = \frac{P^2}{2\epsilon_0}; \; M^r_\theta = M^\theta_z = M^z_\theta = 0 \tag{16}$$

The strain invariants are given by

$$I_1 = \frac{1}{R'} + \frac{r^2}{A^2R^2} + \frac{1}{\lambda^2}\, ; \; I_2 = \frac{r^2}{\lambda^2A^2R^2} + \frac{1}{\lambda^2R'^2} + \frac{r^2}{A^2R^2R'^2}$$

$$I_3 = \frac{r^2}{A^2R^2R'^2\lambda^2}\, ; \; I_4 = \frac{P^2}{R'^2}\, ; \; I_5 = \frac{P^2}{R'^4}\, ; \; I_6 = P^2 \tag{17}$$

$$\| c_i^{-2k} \| = \begin{bmatrix} 1/R'^4 & 0 & 0 \\ 0 & r^4 / A^4 R^4 & 0 \\ 0 & 0 & 1/\lambda^4 \end{bmatrix}$$

From (9), the local electric field is given by

$$L^{E^r} = \frac{2\lambda A R R' P}{r} \left[\frac{1}{R'^2} \frac{\partial \Sigma}{\partial I_4} + \frac{1}{R'^4} \frac{\partial \Sigma}{\partial I_5} + \frac{\partial \Sigma}{\partial I_6} \right]$$

and the others are identically zero. Field equation (2) is satisfied if

$$\frac{2 \epsilon_0 \lambda A R R' P}{r} \left[\frac{1}{R'} \frac{\partial \Sigma}{\partial I_4} + \frac{1}{R'} \frac{\partial \Sigma}{\partial I_5} + \frac{\partial \Sigma}{\partial I_6} \right] = P \qquad (18)$$

As P is a non-zero factor, (18) is a differential equation in R' and R which can be solved when the form of Σ is known. This equation does not give the polarization vector but leaves it still arbitrary. As $R = R(r)$ is determined from this equation, the deformation tensor depends on the material constants unlike in the problems of incompressible materials[5]. Also, $R = R(r)$ depends on the constants entering through the strain invariants containing the polarization vector. Because $P(r)$ cancels out in (18) leaving a first order differential equation (21), we could consider non-homogeneous radial deformations only when a surface charge is applied to the outer surface.

From (8), we get the non-vanishing local stress components to be

$$L^r_r = \frac{2\lambda A R R'}{r} \left[\frac{1}{R'^2} \frac{\partial \Sigma}{\partial I_1} + \left(\frac{r^2}{A^2 R^2} + \frac{1}{\lambda^2} \right) \frac{1}{R'^2} \frac{\partial \Sigma}{\partial I_2} + \frac{r^2}{A^2 R^2 R'^2 \lambda^2} \frac{\partial \Sigma}{\partial I_3} + \frac{P^2}{R'^2} \frac{\partial \Sigma}{\partial I_4} + \frac{2 P^2}{R'^4} \frac{\partial \Sigma}{\partial I_5} \right]$$

$$L^\theta_\theta = \frac{2\lambda A R R'}{r} \left[\frac{r^2}{A^2 R^2} \frac{\partial \Sigma}{\partial I_1} + \frac{r^2}{A^2 R^2} \left(\frac{1}{R'^2} + \frac{1}{\lambda^2} \right) \frac{\partial \Sigma}{\partial I_2} + \frac{r^2}{A^2 R^2 R'^2 \lambda^2} \frac{\partial \Sigma}{\partial I_3} \right]$$

$$L^z_z = \frac{2\lambda A R R'}{r} \left[\frac{1}{\lambda^2} \frac{\partial \Sigma}{\partial I_1} + \frac{1}{\lambda^2} \left(\frac{1}{R'^2} + \frac{r^2}{A^2 R^2} \right) \frac{\partial \Sigma}{\partial I_2} + \frac{r^2}{A^2 R^2 R'^2 \lambda^2} \frac{\partial \Sigma}{\partial I_3} \right] \qquad (19)$$

The Cauchy stress tensor in (1) can be calculated from (6), (16) and (19). The first of the equations of equalibrium (1) reduces to

$$\frac{d t^r_r}{dr} + \frac{(t^r_r - t^\theta_\theta)}{r} = 0 \qquad (20)$$

as I_r ($r = 1$ to 6) and the strain energy function Σ are functions of r only. The other equations are satisfied identically. Equation (20) gives on integration :

$$p^2 f_2 \exp \left[\int \frac{f_3}{r f_2} dr \right] = D - \int \left(f_1' + \frac{f_4}{r} \right) \exp \left(\int \frac{f_3}{r f_2} dr \right) dr \qquad (21)$$

where

$$f_1(r) = \frac{2\lambda A R}{r R'} \left\{ \frac{\partial \Sigma}{\partial I_1} + \left(\frac{1}{\lambda^2} + \frac{r^2}{A^2 R^2} \right) \frac{\partial \Sigma}{\partial I_2} + \frac{r^2}{A^2 R^2 \lambda^2} \frac{\partial \Sigma}{\partial I_3} \right\}$$

$$f_2(r) = \frac{2\lambda A R}{r R'} \frac{\partial \Sigma}{\partial I_4} + \frac{4\lambda A R}{r R'^3} \frac{\partial \Sigma}{\partial I_5} - \frac{1}{2\epsilon_0}$$

$$f_3(r) = f_2(r) - \frac{1}{2\epsilon_0}$$

$$f_4(r) = \frac{2\lambda A R R'}{r} \left(\frac{1}{R'^2} - \frac{r^2}{A^2 R^2} \right) \left(\frac{\partial \Sigma}{\partial I_1} + \frac{1}{\lambda^2} \frac{\partial \Sigma}{\partial I_2} \right)$$

This determines the polarization undetermined so far. The two constants of integration in (1) and (18) may be determined from the condition

$$t_r^r = 0 \text{ on } r = c_1 \text{ and } r = d_1 \tag{22}$$

Thus the polarization vector on using (21), deformation tensors on using (17) and (18) and stress distribution on using (6), (16) and (19), can be determined in complete. It is interesting to observe that both the transverse stress t_r^r and axial stress t_z^z are independent of the polarization vector $P(r)$ explicitly. Also it is to be pointed that the above parameters are independent of the surface charge density ω_f on the surface $r = c_1$.

Boundary Conditions

To maintain the deformation, we should apply

(*i*) a resultant force F, to the edges $\theta = \pm\,\theta_0$ given by

$$F_1 = \int_{d_1}^{c_1} t_\theta \, dr$$

(*ii*) a resultant force F_2 to the edges $z = \pm\, z_0$ given by

$$F_2 = \int_{d_1}^{c_2} t_z^z \, dr$$

(*iii*) a couple M_1 to the edge $\theta = \pm\,\theta_0$ given by

$$M_1 = \int_{d_1}^{c_1} r \, t_\theta^\theta \, dr$$

(*iv*) a couple M_2 to the edge $z = \pm\, z_0$ given by

$$M_2 = \int_{d_1}^{c_1} t_z^z \, r \, dr$$

A PARTICULAR CASE

To have a clear idea of the above results, let us assume

$$\Sigma = \alpha_1 (I_1 - 3) + \alpha_2 (I_2 - 3) + \alpha_4 I_4 + \alpha_6 I_6 \qquad (23)$$

The local electric field is given by

$$\overset{E_r}{L} = \frac{2\lambda A R R' P}{r} \left[\frac{\alpha_4}{R'^2} + \alpha_6 \right] \qquad (24)$$

and (18) reduces to

$$2\, \epsilon_0 \lambda A R \,(\alpha_4 + \alpha_6 R'^2) = r R' \qquad (25)$$

which gives on integration

$$\left[\alpha_6 t_r + (b\alpha_4 - \alpha_6) r \right]^{b\alpha_4 - \alpha_6} = A_1^{\alpha_6} (t_r + r)^{\alpha_6} \qquad (26)$$

where $b = 4 \epsilon_0 \lambda A \alpha_6$; $b R\sqrt{\alpha_4} = r\sqrt{\alpha_6 (1 - t^2)}$
and A_1 is a constant of integration.

Similarly equation (21) can also be integrated for P^2 and obtained as

$$P^2 = \frac{2\epsilon_0 (\alpha_2 + \alpha_1 \lambda^2)}{\lambda^2} \left[4 \lambda^2 \epsilon_0 \alpha_4 \log \frac{(B-r)^2}{2r-B} - \frac{1}{2\epsilon_0 \alpha_4} \log (Br - r^2) \right]$$

$$+ \frac{1}{\lambda^2 A^2 \alpha_4} \Bigg[A^2 (\alpha_2 + \alpha_1 \lambda^2) \log r + \left\{ \alpha_2 b \lambda^2 + A^2 (\alpha_2 + \alpha_1 \lambda^2) \right\}.$$

$$\log (B - r) - \frac{\alpha_2 b \lambda^2 r}{B} \Bigg] - 2 \epsilon_0 D \qquad (27)$$

where $A_1 B = 1$ and D is a constant of integration. Thus the polarization P^2 and $R = R(r)$ are determined through (26) and (27). The stress distribution and the electric field can be calculated.

CYLINDER UNDER UNIFORM PRESSURE

The results for this can be easily obtained by putting $A = 1$ and $\lambda = 1$ as $\theta_0 = \theta_0 = \pi$ and the displacement with axial direction is not considered. The boundary conditions (22) are to be modified to

$$t^r_r = -p_i \text{ when } r = d_1 \text{ and}$$
$$= - p_e \text{ when } r = c_1 \qquad (28)$$

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