INTERACTION OF OSCILLATORY FLOW WITH A NON-UNIFORMLY ROTATING LAMINA

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The problem of the boundary layer flow near the stagnation point of a lamina rotating unsteadily in the presence of a fluctuating free stream directed normally towards it, has been studied in this paper. The velocity distribution has been obtained for the two limiting cases of large and small values of the frequency of oscillation. The transitional frequencies for which the two approximate solutions overlap have been obtained and presented in a tabulated form.

During the past few decades the study of unsteady laminar boundary layers was restricted to problems of impulsive or oscillatory boundary layer growth on rigid bodies with or without a mean velocity in the flow field. Only recently considerable attention has been given to the problems concerning the effect of a fluctuating free stream flow on the boundary layer growth over a vibrating or oscillating body. This study has many practical applications, e.g. in acoustics, turbomachinery and missile dynamics.

Lighthill[1] initiated the study of the response of the fluctuations present in the main stream to the boundary layer growth on a two dimensional body. He obtained the low and high frequency solutions by using the momentum-integral method. Since then a number of papers[2-19] have appeared on the subject. Recently a survey of the study of the response of the laminar boundary layer to a fluctuating stream has also been made[15]. Srivastava[19] extended Lighthill's work to investigate the axi-symmetric boundary layer fluctuations near the stagnation point.

But in the above problems rotation of the boundary or the rotating flow of the fluid is not taken into consideration. In most of the problems of design, such as, the ship propeller behind the hull, the circumferential or the rotatory flow interacts with the axial flow. A simple mathematical model to study the effect of rigid boundary rotation on the fluid motion was investigated by Theodore Von Kàrmàn. Only steady flow in the absence of an axial flow was considered by him. Schlichting & Truckenbrodt[20] studied the Kàrmàn problem in the presence of axial flow. Further detailed studies of this problem were made by different authors[21,22]. The attempts to investigate the unsteady flow of the fluid induced by the torsional oscillations of a lamina about the axis normal to its plane were made by Rosenblat[23] and Benney[24]. Benney also considered the flow induced by a lamina oscillating about a mean rotation in the fluid which is also rotating with uniform angular velocity at infinity[25]. The solution obtained by Benney is fairly complicated even when the external axial flow is absent. All these investigations in a way do help the designer in making suitable mechanical systems involving the rotation as well as the translatory motion along the axis of rotation. But the actual flow fluctuations of the fluid near the boundary, induced in such a situation, it seems, have not been studied so far.

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In the present paper, we have studied the boundary layer flow in the presence of a fluctuating free stream near the stagnation point of a lamina oscillating in its own plane. The velocity components in the directions of r and z outside the boundary layer are taken as

$$U = (a_0 + a_1 e^{i\omega t}) r, \quad W = -2 (a_0 + a_1 e^{i\omega t}) z$$

respectively, where a_0 represents the mean axial flow, a_1 and ω are the amplitude and the frequency of oscillation. The time dependent angular velocity of the disc has also been taken of the similar form

$$\Omega = (\Omega_0 + \Omega_1 e^{i\omega t})$$

where Ω_0 represents the mean rotation and Ω_1 and ω are the amplitude and frequency of the secondary flow. Both oscillatory components of the axial and rotational velocity are assumed to be of the same frequency.

Taking the amplitudes of the oscillatory components a_1 and Ω_1 to be small (retaining the first order terms in a_1 and Ω_1) two situations have been considered. In one case $\Omega_0 \gg a_0$ and in the other $a_0 \gg \Omega_0$. The former corresponds to the case when the mean circumferential flow is dominant and the later to the case when the mean axial flow is dominant. When the angular velocity of the disc is zero, the problem reduces to that of axi-symmetric unsteady stagnation point flow studied earlier by Srivastava[19]. The solution of the unsteady flow problem presented in this paper is the first order perturbation due to a_1 and Ω_1 on the steady state solution obtained by Schlichting & Truckenbrodt[20]. The solution is obtained numerically for low and high values of frequency. Also the numerical values of the transitional frequencies for which the low and high frequency solutions overlap are presented in a tabulated form.

FORMULATION OF THE PROBLEM

Let us consider that a fluctuating fluid impinges normally on a lamina rotating in its own plane. The resulting flow of the viscous incompressible fluid in this case is axially-symmetric and is governed by the equations

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} - \frac{v^2}{r} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left\{ \frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r} \left(\frac{u}{r} \right) + \frac{\partial^2 u}{\partial z^2} \right\} \quad (1)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + \frac{uv}{r} + w \frac{\partial v}{\partial z} = \nu \left\{ \frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r} \left(\frac{v}{r} \right) + \frac{\partial^2 v}{\partial z^2} \right\} \quad (2)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left\{ \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \right\} \quad (3)$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \quad (4)$$

where all symbols have their usual meanings.

The lamina is represented by the plane $z = 0$ and the fluid occupies the region $z > 0$. Taking the stagnation point as the origin, the velocity components in the directions of r and z just outside the boundary layer can be written as

$$U = ar, \quad W = -2az, \text{ where } a = (a_0 + a_1 e^{i\omega t}) \quad (5)$$

Thus the boundary conditions are

$$\left.\begin{aligned} & u = 0, v = r\,(\Omega_0 + \Omega_1\, e^{i\omega t}),\ w = 0 \text{ at } z = 0 \\ & u = r\,(a_0 + a_1\, e^{i\omega t}),\ v = 0 \text{ at } z = \delta \end{aligned}\right\} \quad (6)$$

where δ is the boundary layer thickness.

The velocity components within the boundary layer in the neighbourhood of the axis of rotation can be taken as

$$u = rf\,(z, t),\ v = rg\,(z, t),\ w = h\,(z, t) \quad (7)$$

We also assume that the velocity components and the pressure within the boundary layer vary sinusoidally about steady mean values and write

$$f(z,t) = F_0(\eta) + F_1(\eta)\, e^{i\omega t} \quad (8)$$

$$g(z,t) = G_0(\eta) + G_1(\eta)\, e^{i\omega t} \quad (9)$$

$$h(z,t) = H_0(\eta) + H_1(\eta)\, e^{i\omega t} \quad (10)$$

$$p(r,z,t) = P_0(r,z) + P_1(r,z)\, e^{i\omega t} \quad (11)$$

where F_1, G_1 and H_1 are small and $\eta = z/\delta$.

The boundary conditions (6) now become

$$\left.\begin{aligned} & F_0 = F_1 = 0,\ G_0 = \Omega_0,\ G_1 = \Omega_1,\ H_0 = H_1 = 0 \text{ at } \eta = 0 \\ & F_0 = a_0,\ F_1 = a_1,\ F_0' = F_0'' = F_1' = F_1'' = 0 \\ & G_0 = G_1 = 0,\ G_0' = G_0'' = G_1' = G_1'' = 0 \text{ at } \eta = 1 \end{aligned}\right\} \quad (12)$$

Substituting (8) to (11) in equations (1) to (4) and equating the time independent terms and those which are linear in F_1 G_1, H_1, and their derivatives to zero, we get

$$F_0^2 - G_0^2 + H_0 F_0' \frac{1}{\delta} = -\frac{1}{\rho r}\frac{\partial P_0(r,z)}{\partial r} + (\nu/\delta^2)\, F_0'' \quad (13)$$

$$2F_0 G_0 + H_0 G_0' \frac{1}{\delta} = (\nu/\delta^2)\, G_0'' \quad (14)$$

$$H_0 H_0' = -\frac{1}{\rho}\frac{\partial P_0(r,z)}{\partial \eta} + (\nu/\delta)\, H_0'' \quad (15)$$

$$2F_0 + H_0' \frac{1}{\delta} = 0 \quad (16)$$

and

$$i\omega F_1 + 2\,(F_0 F_1 - G_0 G_1) + (H_0 F_1' + H_1 F_0')\frac{1}{\delta} = -\frac{1}{\rho r}\frac{\partial P_1(r,z)}{\partial r} + (\nu/\delta^2)\, F_1'' \quad (17)$$

$$i\omega G_1 + 2\,(F_0 G_1 + F_1 G_0) + (H_0 G_1' + H_1 G_0')\frac{1}{\delta} = (\nu/\delta^2)\, G_1'' \quad (18)$$

$$i\omega H_1 + (H_0 H_1' + H_1 H_0') \frac{1}{\delta} = - \frac{1}{\rho\delta} \frac{\partial P_1(r,z)}{\partial \eta} + (\nu/\delta^2) H_1'' \qquad (19)$$

$$2 F_1 + H_1' \frac{1}{\delta} = 0 \qquad (20)$$

From (13) and (15) we find that $- \frac{1}{\rho r} \frac{\partial P_0(r,z)}{\partial r}$ = Constant = A_1 say this gives $F_0^2 - G_0^2 + H_0 F_0' \frac{1}{\delta} - (\nu/\delta^2) F_0'' = A_1$.

Using the boundary conditions (12) at $\eta = 1$, one gets $A_1 = a_0^2$ and thus the above expression becomes

$$F_0^2 - G_0^2 + H_0 F_0' \frac{1}{\delta} = a_0^2 + (\nu/\delta^2) F_0'' \qquad (21)$$

Similarly from (17) and (19) $- \frac{1}{\rho r} \frac{\partial P_1(r,z)}{\partial r} = A_2$ where A_2 is a constant.

The boundary condition at $\eta = 1$ gives $A_2 = 2a_0 a_1 + i\omega a_1$ and thus we get

$$i\omega (F_1 - a_1) + 2 (F_0 F_1 - G_0 G_1) + (H_0 F_1' + H_1 F_0') \frac{1}{\delta} = 2a_0 a_1 + (\nu/\delta^2) F_1'' \qquad (22)$$

The equations (21), (14) to (16) are those derived by Schlichting & Truckenbrodt[20] for the steady flow of fluid impinging on a disc rotating with uniform angular velocity. The functions $F_0(\eta)$ and $G_0(\eta)$ in that case are given by

$$F_0(\eta) = a_0\eta^3 (10 - 15\eta + 6\eta^2) + c_1\eta (1 - 6\eta^2 + 8\eta^3 - 3\eta^4) + c_2\eta^2 (1 - 3\eta + 3\eta^2 - \eta^3), \qquad (23)$$

and

$$G_0(\eta) = \Omega_0 (1 - 2\eta + 2\eta^3 - \eta^4), \qquad (24)$$

where c_1 and c_2 are constants.

Two cases (*i*) $\Omega_0 \leqslant a_0$ and (*ii*) $\Omega_0 \geqslant a_0$ have been discussed by them.

(*i*) When $\Omega_0 \leqslant a_0$ and $q = \Omega_0/a_0$.

Dividing the expressions (23) and (24) by a_0, one gets

$$\frac{F_0}{a_0} = \eta^3 (10 - 15\eta + 6\eta^2) + \frac{c_1}{a_0} \eta (1 - 6\eta^2 + 8\eta^3 - 3\eta^4) + \frac{c_2}{a_0} \eta^2 (1 - 3\eta + 3\eta^2 - \eta^3)$$

and $\frac{G_0}{a_0} = q (1 - 2\eta + 2\eta^3 - \eta^4)$

The values of the constants c_1/a_0, c_2/a_0, $(a_0/\nu)\delta^2$ for different values of q as obtained by Schlichting & Truckenbrodt[20] are given in Table 1.

(*ii*) When $\Omega_0 \geqslant a_0$ and $p = a_0/\Omega_0$. Now dividing (23) and (24) by Ω_0, one gets

$$F_0/\Omega_0 = p\eta^3 (10 - 15\eta + 6\eta^2) + c_1/\Omega_0\eta (1 - 6\eta^2 + 8\eta^3 - 3\eta^4) + c_2/\Omega_0\eta^2 (1 - 3\eta + 3\eta^2 - \eta^3)$$

and $G_0/\Omega_0 = (1 - 2\eta + 2\eta^3 - \eta^4)$

The constants c_1/Ω_0, c_2/Ω_0 and $(\Omega_0/\nu)\delta^2$ for different values of p are given in Table 2.

TABLE 1

VALUES OF CONSTANTS c_1/a_0, c_2/a_0 and $a_0\delta_0/\nu$ FOR DIFFERENT VALUES OF q

q	c_1/a_0	c_2/a_0	$a_0\delta^2/\nu$
1	2·961	—3·730	3·73
$\frac{3}{4}$	2·764	—2·992	3·83
$\frac{1}{2}$	2·625	—2·438	3·90
$\frac{1}{4}$	2·535	—2·098	3·95
0	2·502	—1·985	3·97

TABLE 2

VALUES OF CONSTANTS c_1/Ω_0, c_2/Ω_0 AND FOR DIFFERENT VALUES OF p

p	c_1/Ω_0	c_2/Ω_0	$\Omega_0\delta^2/\nu$
0	1·929	—6·535	13·07
$\frac{1}{4}$	1·832	—4·978	9·37
$\frac{1}{2}$	2·054	—4·050	6·48
$\frac{3}{4}$	2·454	—3·742	4·79
1	2·961	—3·730	3·73

ANALYSIS

It is not easy to get the exact solutions of the equations (18) to (20) and (22) inspite of their being linear in F_1, G_1 and H_1 for any arbitrary value of the frequency. Therefore we obtain a solution for the low frequency range by using the idea of quasi-steady state which has been introduced by Lighthill[1] to study the two dimensional boundary layer flow in a fluctuating free stream. For the high frequency solution we retain only the terms having ω as the co-efficient and the second order derivatives of F_1, G_1 and H_1. It is justified on the basis of singular perturbation theory used for the solution of a differential equation when the co-efficient of its highest derivative is a vanishingly small quantity.

(a) *Low Frequency*

The solutions of equations (22) and (18) to (20) in the limiting case when $\omega \to 0$ are the quasi-steady solutions. Let them be denoted by $F_2(\eta)$, $G_2(\eta)$ and $H_2(\eta)$. The differential equations (22) and (18) to (20) for $\omega = 0$ become

$$2(F_0 F_2 - G_0 G_2) + (H_0 F_2' + H_2 F_0')\frac{1}{\delta} = 2a_0\, a_1 + (\nu/\delta^2)\, F_2'' \tag{25}$$

$$2(F_0 G_2 + F_2 G_0) + (H_0 G_2' + H_2 G_0')\frac{1}{\delta} = (\nu/\delta^2)\, G_2'' \tag{26}$$

$$(H_0 H_2' + H_2 H_0')\frac{1}{\delta} = -\frac{1}{\rho\delta}\frac{\partial P_2(r, z)}{\partial \eta} + (\nu/\delta^2)\, H_2'' \tag{27}$$

$$2F_2 + H_2'\frac{1}{\delta} = 0 \tag{28}$$

The boundary conditions are

$$\left.\begin{aligned} &F_2=0,\quad G_2=\Omega_1,\quad H_2=0 \quad \text{at} \quad \eta=0 \\ &F_2=a_1,\ F_2'=F_2''=0,\ G_2=0,\ G_2'=G_2''=0 \text{ at } \eta=1 \end{aligned}\right\} \tag{29}$$

The equations (25) to (28) are solved by Kàrmàn—Pohlhausen method. Assuming $F_2(\eta)$ and $G_2(\eta)$ each to be polynomial of 5th degree and satisfying the boundary conditions (29), we have

$$F_2(\eta)=a_1\eta^3(10-15\eta+6\eta^2)+K_1\eta(1-6\eta^2+8\eta^3-3\eta^4)+K_2\eta^2(1-3\eta+3\eta^2-\eta^3) \tag{30}$$

and
$$G_2(\eta)=\Omega_1(1-10\eta^3+15\eta^4-6\eta^5)+B\eta(1-6\eta^2+8\eta^3-3\eta^4) \tag{31}$$

The co-efficients K_2 and B satisfy the relations

$$K_2=-\delta^2/\nu(\Omega_0\Omega_1+a_0a_1)$$

and
$$B=\left(\frac{\partial G_2}{\partial\eta}\right)_0$$

Integrating (25) and (26) between the limits 0 to 1 and using (28) we get

$$-(\nu/\delta^2)F_2'\Big|_{\eta=0}=2\int_0^1(3F_0F_2-G_0G_2)\,d\eta-2\int_0^1(a_1F_0+a_0F_1)\,d\eta-2a_0a_1 \tag{32}$$

and
$$-(\nu/\delta^2)G_2'\Big|_{\eta=0}=4\int_0^1(F_0G_2+F_2G_0)\,d\eta \tag{33}$$

Substituting the expressions for F_0, F_2, G_0, G_2 etc. into (32) and (33) we get

$$\left.\begin{aligned} &\left(\frac{\nu}{a_0\delta^2}+\frac{151}{1155}+\frac{208}{3465}\frac{c_1}{a_0}+\frac{23}{2310}\frac{c_2}{a_0}\right)B+\left(\frac{47}{315}q\right)K_1=-\left[\left(\frac{100}{231}+\frac{311}{1155}\frac{c_1}{a_0}+\frac{281c_2}{6930a_0}\right)\right. \\ &\qquad\left.-\frac{13}{630}\frac{a_0\delta^2}{\nu}q^2\right]\Omega_1-\left(\frac{6}{35}-\frac{13}{630}a_0\delta^2/\nu\right)qa_1 \\ &\left(\frac{\nu}{a_0\delta^2}-\frac{3}{770}+\frac{104}{1155}\frac{c_1}{a_0}+\frac{23}{1540}\frac{c_2}{a_0}\right)K_1-\left(\frac{47}{630}q\right)B=\left[\left(\frac{127}{77}+\frac{3}{770}\frac{c_1}{a_0}-\frac{27}{4620}\frac{c_2}{a_0}\right)\right. \\ &\left.+\left(\frac{27}{4620}+\frac{23}{1540}\frac{c_1}{a_0}+\frac{1}{385}\frac{c_2}{a_0}\right)\frac{a_0}{\nu}^2\right]a_1+\left[\left(\frac{27}{4620}+\frac{23}{1540}\frac{c_1}{a_0}+\frac{1}{385}\frac{c_2}{a_0}\right)\frac{a_0\delta^2}{\nu}+\frac{18}{35}\right]q\Omega_1 \end{aligned}\right\} 34$$

for the case $\Omega_0 < a_0$

and

$$\left.\begin{aligned} &\left(\frac{\nu}{\Omega_0\delta^2}+\frac{151}{1155}p+\frac{208}{3465}\frac{c_1}{\Omega_0}+\frac{23}{2310}\frac{c_2}{\Omega_0}\right)B+\left(\frac{47}{315}\right)K_1= \\ &-\left[\left(\frac{100}{231}p+\frac{311}{1155}\frac{c_1}{\Omega_0}+\frac{281}{6930}\frac{c_2}{\Omega_0}\right)-\frac{13}{630}\Omega_0\delta^2/\nu\right]\Omega_1-\left(\frac{6}{35}-\frac{13}{630}p\,\Omega_0\delta^2/\nu\right)a_1 \\ &\left(\frac{\nu}{\Omega_0\delta^2}-\frac{3}{770}p+\frac{104}{1155}\frac{c_1}{\Omega_0}+\frac{23}{1540}\frac{c_2}{\Omega_0}\right)K_1-\left(\frac{47}{630}\right)B=\left[\left(\frac{127}{77}p+\frac{3}{770}\frac{c_1}{\Omega_0}-\frac{27}{4620}\frac{c_2}{\Omega_0}\right)\right. \\ &\left.+\left(\frac{27}{4620}p+\frac{23}{1540}\frac{c_1}{\Omega_0}+\frac{1}{385}\frac{c_2}{\Omega_0}\right)\frac{\Omega_0\delta^2}{\nu}p\right]a_1+\left[\left(\frac{27}{4620}p+\frac{23}{1540}\frac{c_1}{\Omega_0}+\frac{1}{385}\frac{c_2}{\Omega_0}\right)\Omega_0\delta^2/\nu\right. \\ &\qquad\left.+\frac{18}{35}\right]\Omega_1 \end{aligned}\right\} \tag{35}$$

for $\Omega_0 \geqslant a_0$.

TABLE 3

NUMERICAL VALUES OF THE CONSTANTS B AND K_1 FOR CASE (i) $\Omega_0 \leqslant a_0$

$q = \Omega_0/a_0$, $\partial = \Omega_1/a_2$, $\mu = a_1/\Omega_1$

	q / λ		1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$\Omega_1 \leqslant a_1$	1	$-B/a_1$	3·34219	2·80431	2·26643	1·72855	1·19067
		K_1/a_1	4·73334	4·46790	4·20249	3·93706	3·67164
	$\frac{3}{4}$	$-B/a_1$	3·03816	2·51534	1·99252	1·46970	·94688
		K_1/a_1	4·67530	4·46845	4·26163	4·05477	3·84793
	$\frac{1}{2}$	$-B/a_1$	2·70078	2·19001	1·67927	1·16851	·65776
		K_1/a_1	4·54551	4·40381	4·26211	4·12042	3·97872
	$\frac{1}{4}$	$-B/a_1$	2·35008	1·84721	1·34433	·84145	·33857
		K_1/a_1	4·35969	4·28751	4·21530	4·14311	4·07091
	0	$-B/a_1$	2·00031	1·50024	1·00016	·50000	·00000
		K_1/a_1	4·10667	4·10667	4·10667	4·10667	4·10667

	q / μ		0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Omega_1 \geqslant a_1$	1	$-B/\Omega_1$	2·15152	2·44919	2·74686	3·04453	3·34219
		K_1/Ω_1	1·06170	1·97961	2·89752	3·81543	4·73334
	$\frac{3}{4}$	$-B/\Omega_1$	2·09128	2·32799	2·56470	2·80141	3·03816
		K_1/Ω_1	·82737	1·78935	2·75133	3·61331	4·67530
	$\frac{1}{2}$	$-B/\Omega_1$	2·04302	2·20746	2·37190	2·53634	2·70078
		K_1/Ω_1	·56679	1·56147	2·55615	3·55083	4·54551
	$\frac{1}{4}$	$-B/\Omega_1$	2·01151	2·09616	2·18080	2·25546	2·35008
		K_1/Ω_1	·28878	1·30651	2·32424	3·34197	4·35969
	0	$-B/\Omega_1$	2·00031	2·00031	2·00031	2·00031	2·00031
		K_1/Ω_1	·00000	1·02667	2·05334	3·08001	4·10667

Now for each case (i) $\Omega_0 \leqslant a_0$ and (ii) $\Omega_0 \geqslant a_0$ two more sub-cases arise according as $\Omega_1 \leqslant a_1$ or $\Omega_1 \geqslant a_1$. When $\Omega_1 \leqslant a_1$ the ratio $\Omega_1/a_1 = \lambda$ whereas for the other case $a_1/\Omega_1 = \mu$.

The numerical values of the constants B and K_1, for the two cases (i) $\Omega_0 < a_0$ and (ii) $\Omega_0 \geqslant a_0$ are given in the Tables 3 and 4.

For general values of ω we may write

$$\left.\begin{aligned} F_1(\eta) &= F_2(\eta) + i\omega F_3(\eta) \\ G_1(\eta) &= G_2(\eta) + i\omega G_3(\eta) \\ H_1(\eta) &= H_2(\eta) + i\omega H_3(\eta) \end{aligned}\right\} \quad (36)$$

Substituting (36) into (22), (18) and (20) and noting that F_2, G_2 and H_2 are the solutions of these equations for $\omega = 0$, we get

TABLE 4

NUMERICAL VALUES OF THE CONSTANTS B AND K_1 FOR CASE (*ii*) $\Omega_0 \geqslant a_0$

$p = a_0 / \Omega_0$, $\lambda = \Omega_1 / a_1$, $\mu = a_1 / \Omega_1$

	$p \backslash \lambda$		1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$\Omega_1 \leqslant a_1$	0	$-B/a_1$	4·27127	3·47299	2·67471	1·87643	1·07815
		K_1/a_1	2·59311	1·88743	1·18172	·47603	—·22967
	$\frac{1}{4}$	$-B/a_1$	4·68991	3·99704	3·30420	2·61134	1·91849
		K_1/a_1	4·07045	3·48884	2·90723	2·32562	1·74401
	$\frac{1}{2}$	$-B/a_1$	4·18853	3·58351	2·97849	2·37347	1·76845
		K_1/a_1	4·63352	4·19671	3·75993	3·32313	2·88634
	$\frac{3}{4}$	$-B/a_1$	3·70380	3·14304	2·58231	2·02156	1·46082
		K_1/a_1	4·76908	4·43414	4·09919	3·76424	3·42929
	1	$-B/a_1$	3·34219	2·80431	2·26643	1·72855	1·19067
		K_1/a_1	4·73324	4·46790	4·20249	3·93706	3·67164

	$p \backslash \mu$		0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Omega_1 \geqslant a_1$	0	$-B/\Omega_1$	3·19312	3·46266	3·73220	4·00174	4·27127
		K_1/Ω_1	2·82278	2·76537	2·70795	2·65055	2·59311
	$\frac{1}{4}$	$-B/\Omega_1$	2·77142	3·25104	3·73066	4·21028	4·68991
		K_1/Ω_1	2·32644	2·76244	3·19844	3·63444	4·07045
	$\frac{1}{2}$	$-B/\Omega_1$	2·42008	2·86219	3·30430	3·74641	4·18853
		K_1/Ω_1	1·74718	2·46876	3·19035	3·91192	4·63352
	$\frac{3}{4}$	$-B/\Omega_1$	2·24298	2·60819	2·97339	3·33861	3·70380
		K_1/Ω_1	1·33979	2·19711	3·05443	3·91175	4·76908
	1	$-B/\Omega_1$	2·15152	2·44919	2·74686	3·04453	3·34219
		K_1/Ω_1	1·06170	1·97961	2·89752	3·81543	4·73334

$$\left(F_2 - a_1\right) + 2F_0F_3 - 2G_0G_3 + \left(H_0F_3' + H_3F_0'\right)\frac{1}{\delta} = (\nu/\delta^2)\, F_3'' \qquad (37)$$

$$G_2 + 2\left(F_0G_3 + F_3G_0\right) + \left(H_0G_3' + H_3G_0'\right)\frac{1}{\delta} = (\nu/\delta^2)\, G_3'' \qquad (38)$$

$$2F_3 + H_3'\,\frac{1}{\delta} = 0 \qquad (39)$$

Boundary conditions on F_3 , G_3 and H_3 are

$$\left.\begin{array}{ll} F_3 = G_3 = H_3 = 0\,,\ F_3'' = -\,a_1\delta^2/\nu\,,\ G_3'' = \Omega_1\delta^2/\nu & \text{at } \eta = 0 \\ F_3 = F_3' = F_3'' = 0\,,\ G_3 = G_3' = G_3'' = 0 & \text{at } \eta = 1 \end{array}\right\} \qquad (40)$$

We again solve the equations (37) to (39) under the boundary conditions (40) by Kârmân-Pohlhausen method. We assume F_3 and G_3 each to be plynomial of 5th degree and write

$$F_3 = A\eta\left(1 - 6\eta^2 + 8\eta^3 - 3\eta^4\right) - \frac{1}{2}a_1\delta^2/\nu\left(1 - 3\eta + 3\eta^2 - \eta^3\right)\eta^2 \quad (41)$$

and

$$G_3 = E\eta\left(1 - 6\eta^2 + 8\eta^3 - 3\eta^4\right) + \frac{1}{2}\Omega_1\delta^2/\nu\left(1 - 3\eta + 3\eta^2 - \eta^3\right)\eta^2 \quad (42)$$

These expressions are taken to satisfy the conditions (40).
Integrating the equations (37) and (38) between the limits $\eta = 0$ to $\eta = 1$ and using (39) we get

$$-\left(\nu/\delta^2\right)F_3'\Big|_{\eta=0} = \int_0^1 (F_2 - a_1)\,d\eta - 2\int_0^1 a_0F_3\,d\eta + 2\int_0^1 (3F_0F_3 - G_0G_3)\,d\eta \quad (43)$$

and

$$-(\nu/\delta^2)\,G_3'\Big|_{\eta=0} = \int_0^1 G_2\,d\eta + 4\int_0^1 (F_0G_3 + F_3G_0)\,d\eta \quad (44)$$

Substituting the expressions for $F_0, F_2, F_3, G_0, G_2, G_3$ etc. into (43) and (44) and arranging the terms we get two simultaneous algebraic equations in E and A. These equations are solved for E and A in the two cases $\Omega_0 \leqslant a_0$ and $\Omega_0 \geqslant a_0$ and the sub-cases $\Omega_1 \leqslant a_1$ and $\Omega_1 \geqslant a_1$. The numerical values of the constants E and A are given in Tables 5 and 6.

b) High Frequency

For high frequency, i.e. when ω is greater than some as yet undetermined value, the above treatment will not give a correct picture. For this case, we approximate the equations (17) and (18) by retaining terms involving ω and the derivative of the highest order. The equations (22) and (18) are then reduced to

$$i\omega\,(F_1 - a_1) = (\nu/\delta^2)\,F_1'' \quad (45)$$

$$i\omega\,G_1 = (\nu/\delta^2)\,G_1'' \quad (46)$$

The boundary conditions on F_1 and G_1 are

$$\left.\begin{aligned} F_1 &= 0, \quad G_1 = \Omega_1 \quad \text{at} \quad \eta = 0 \\ F_1 &= a_1, \quad G_1 = 0 \quad \text{at} \quad \eta = 1 \end{aligned}\right\} \quad (47)$$

The solutions of (45) and (46) are

$$F_1 = a_1\left[1 + e^{-(1+i)\left(\frac{\omega}{2\nu}\right)^{\frac{1}{2}}\delta\eta}\right] \quad (48)$$

and

$$G_1 = \Omega_1\, e^{-(1+i)\left(\frac{\omega}{2\nu}\right)^{\frac{1}{2}}\delta\eta} \quad (49)$$

This solution shows that in the higher frequency range the velocity components within the boundary layer oscillate with respect to z also due to viscosity and remain unaffected by the mean flow,

TABLE 5

NUMERICAL VALUES OF THE CONSTANTS E AND A FOR CASE (*i*) $\Omega_0 \leqslant a_0$

$$q = \Omega_0 / a_0, \quad \lambda = \Omega_1 / a_1, \quad \mu = a_1 / \Omega_1$$

	$q \backslash \lambda$		1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
	1	$-E'$	·52607	·36229	·19850	·03472	—·12907
		A'	·43434	·47309	·51182	·55057	·58930
	$\frac{3}{4}$	$-E'$	·58388	·40895	·23412	·05919	—·11564
		A'	·43716	·46923	·50128	·53336	·56536
$\Omega_1 \leqslant a_1$	$\frac{1}{2}$	$-E'$	·64780	·46405	·28031	·09656	—·08716
		A'	·45168	·47462	·49755	·52040	·54345
	$\frac{1}{4}$	$-E'$	·71191	·52210	·33229	·14247	—·04734
		A'	·47904	·49107	·50310	·51514	·52718
	0	$-E'$	·76885	·57663	·38442	·19222	·00000
		A'	·52085	·52085	·52085	·52085	·52085

where $E' = E\, a_0 / a_1$ and $A' = A\, a_0 / a_1$

	$q \backslash \mu$		0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
	1	$-E''$	·65514	·62287	·59059	·55833	·52607
		A''	—·15496	—·00764	·13969	·28701	·43434
	$\frac{3}{4}$	$-E''$	·69952	·67062	·64172	·61728	·58388
		A''	—·12826	·01310	·15446	·31695	·43716
$\Omega_1 \geqslant a_1$	$\frac{1}{2}$	$-E''$	·73497	·71317	·69138	·66959	·64780
		A''	—·09177	·04409	·17995	·31582	·45168
	$\frac{1}{4}$	$-E''$	·75924	·74740	·73556	·72567	·71191
		A''	—·04814	·08366	·21545	·34716	·47904
	0	$-E''$	·76885	·76885	·76885	·76885	·76885
		A''	·00000	·13021	·26043	·39063	·52085

Where $E'' = E\, a_0/\Omega_1$ and $A'' = A a_0/ \Omega_1$

TABLE 6

NUMERICAL VALUES OF THE CONSTANTS E AND A FOR CASE (*ii*) $\Omega_0 \geqslant a_0$

$p = a_0 / \Omega_0$, $\lambda = \Omega_1 / a_1$, $\mu = a_1 / \Omega_1$

	p / λ		1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$\Omega_1 \leqslant a_1$	0	$-\bar{E}'$	2·89553	2·60168	2·30986	2·01403	1·72020
		$\bar{A}'$	2·53842	2·67312	2·81582	2·95452	3·09322
	$\frac{1}{4}$	$-\bar{E}'$	1·25698	·98685	·71651	·44628	·17605
		$\bar{A}'$	1·61571	1·75420	1·89266	2·03104	2·16960
	$\frac{1}{2}$	$-\bar{E}'$	·77454	·53677	·29922	·06157	—·17608
		$\bar{A}'$	·90000	·99410	1·08819	1·18228	1·27638
	$\frac{3}{4}$	$-\bar{E}'$	·61704	·41909	·22116	·02322	—·17471
		$\bar{A}'$	·58374	·64336	·70299	·76264	·82228
	1	$-\bar{E}'$	·52607	·36229	·19850	·03472	—·12907
		$\bar{A}'$	·43434	·47306	·51182	·55057	·58930

where $\bar{E}' = E\,\Omega_0 / a_1$ and $A' = A\,\Omega_0 / a_1$

	p / μ		0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Omega_1 \geqslant a_1$	0	$-\bar{E}''$	1·17557	1·60538	2·03647	2·46548	2·89553
		$\bar{A}''$	—·55480	·21849	·99181	1·76510	2·53842
	$\frac{1}{4}$	$-\bar{E}''$	1·08093	1·12495	1·16895	1·21299	1·25698
		$\bar{A}''$	—·55390	—·01150	·53091	1·07232	1·61571
	$\frac{1}{2}$	$-\bar{E}''$	0·95064	·90662	·86259	·81858	·77455
		$\bar{A}''$	—·37637	—·05727	·26181	·58091	·90000
	$\frac{3}{4}$	$-\bar{E}''$	·79178	·74804	·70440	·66071	·61704
		$\bar{A}''$	—·23855	—·03299	·17259	·37815	·68374
		$-\bar{E}''$	·65513	·62287	·59059	·55833	·52607
		$\bar{A}''$	—·15496	—·00764	·13969	·28701	·43434

where $\bar{E}'' = E\,\Omega_0 / \Omega_1$ and $\bar{A}'' = A\,\Omega_0 / \Omega_1$

DISCUSSION

The expressions for the skin-friction at the wall in the radial and azimuthal directions are given by

$$\bar{\mu}\left(\frac{\partial u}{\partial z}\right)_0 \text{ and } \bar{\mu}\left(\frac{\partial v}{\partial z}\right)_0$$

In the case of high frequency they may be written as

$$\bar{\mu}\,(r/\delta)\left[c_1+\sqrt{\frac{\omega}{2\nu}}\,\delta a_1(1+i)\,e^{iwt}\right] \tag{50}$$

and

$$-\bar{\mu}\,(r/\delta)\left[2\Omega_0+\sqrt{\frac{\omega}{2\nu}}\,\delta\Omega_1\,(1+i)\,e^{iwt}\right] \tag{51}$$

The amplitude of the fluctuation increases with ω and its phase is ahead of the fluctuation of the main stream by 45° in both the directions. In the case of low frequency, expressions for the skin friction in the two directions are :

$$\bar{\mu}\,(r/\delta)\left[c_1+(K_1+iA\omega)\,e^{iwt}\right] \tag{52}$$

and

$$-\bar{\mu}\,(r/\delta)\left[2\Omega_0-(B_1+iE\omega)\,e^{iwt}\right] \tag{53}$$

The skin friction, in the radial and azimuthal direction, has phase lead of $\tan^{-1}(A\omega/K_1)$ and $\tan^{-1}(E\omega/B)$ respectively over the oscillation of the main flow. The phase lead increases with ω and becomes 45° for $\omega=\omega_R=K_1/A$ and $\omega=\omega_A=B/E$ where ω_R and ω_A are the frequencies in the radial and azimuthal directions. Such values of the frequencies ω_R and ω_A for the two cases (*i*) $\Omega_0\leqslant a_0$ and (*ii*) $\Omega_0\geqslant a_0$ and sub-cases $\Omega_1\leqslant a_1$ and $\Omega_1\geqslant a_1$ can easily be determined. When $\Omega_1\leqslant a_1$ we can write

$$\omega_R=\frac{K_1/a_1}{A/a_1} \text{ and } \omega_A=\frac{B/a_1}{E/a_1}$$

and when $\Omega_1\geqslant a_1$ we put

$$\omega_R=\frac{K_1/\Omega_1}{A/\Omega_1} \text{ and } \omega_A=\frac{B/\Omega_1}{E/\Omega_1}$$

Putting the values of the constants K_1, A, B and E for different values of q and p and for λ and μ we determine values of the frequencies ω_R and ω_A given in Tables 7 and 8.

TABLE 7

VALUES OF TRANSITIONAL FREQUENCIES ω_R AND ω_A FOR $\Omega_0 \lesseqgtr a_0$

$q = \Omega_0 / a_0, \quad \lambda = \Omega_1 / a_1, \quad \mu = a_1 / \Omega_1$

	$q \backslash \lambda$		1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$\Omega_1 \leqslant a_1$	1	$\omega R/a_0$	10·89778	9·44408	8·21087	7·15088	6·23051
		$\omega A/a_0$	6·35313	7·74051	11·41778	—	—
	$\frac{3}{4}$	$\omega R/a_0$	10·68835	9·52294	8·50150	7·60231	6·80616
		$\omega A/a_0$	5·20340	6·15073	8·51068	24·8302	—
	$\frac{1}{2}$	$\omega R/a_0$	10·06356	9·27860	8·56619	7·91875	7·32123
		$\omega A/a_0$	4·16916	4·71934	5·99076	12·10138	—
	$\frac{1}{4}$	$\omega R/a_0$	9·10089	8·73095	8·37865	8·04269	7·72205
		$\omega A/a_0$	3·30109	3·53804	4·04564	5·90620	—
	0	$\omega R/a_0$	7·88455	7·88455	7·88455	7·88455	7·88455
		$\omega A/a_0$	2·60170	2·60170	2·60170	2·60170	—
	$q \backslash \mu$		0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Omega_1 \geqslant a_1$	1	$\omega R/a_0$	—	—	20·74250	13·29372	10·89778
		$\omega A/a_0$	3·28406	3·93210	4·57353	5·45292	6·35313
	$\frac{3}{4}$	$\omega R/a_0$	—	—	17·81257	11·40025	10·68835
		$\omega A/a_0$	2·98959	3·47140	3·99660	4·53831	5·20340
	$\frac{1}{2}$	$\omega R/a_0$	—	—	14·20478	11·24321	10·06356
		$\omega A/a_0$	2·77973	3·09528	3·43067	3·78790	4·16916
	$\frac{1}{4}$	$\omega R/a_0$	—	15·61690	10·78784	9·62660	9·10089
		$\omega A/a_0$	2·64937	2·80460	2·96482	3·10811	3·30109
	0	$\omega R/a_0$	—	7·88455	7·88455	7·88455	7·88455
		$\omega A/a_0$	2·60170	2·60170	2·60170	2·60170	2·60170

Dashes (—) represent cases in which smooth transition does not take place.

For these tabulated values of ω_R and ω_A we find that the amplitudes of the oscillation in both the asymptotic cases of high and low frequency are approximately the same. Both the phase and amplitude of the skin-friction fluctuations (high as well as low frequency) are in agreement with these values of ω_R and ω_A. Hence they may reasonably be taken as the values at which transition from one type of flow to the other occurs. The dashes in Tables 7 and 8 represent those cases in which smooth transition does not take place.

In the presence of rotation the numerical results given in Table 7 ($\Omega_1 \leqslant a_1$) show that there exists a value of ω at which the smooth transition from the low to high frequency

TABLE 8

VALUES OF TRANSITIONAL FREQUENCIES ω_R AND ω_A FOR $\Omega_0 \geqslant a_0$

$p = a_0/\Omega_0$, $\lambda = \Omega_1/a_1$, $\mu = a_1/\Omega_1$

	p \ λ		1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$\Omega_1 \leqslant a_1$	0	$\omega R/\Omega_0$	1·02154	·70608	—	—	—
		$\omega A/\Omega_0$	—	—	—	—	—
	$\frac{1}{4}$	$\omega R/\Omega_0$	2·51929	1·98885	1·53606	1·14504	·80384
		$\omega A/\Omega_0$	3·73109	4·05030	—	—	—
	$\frac{1}{2}$	$\omega R/\Omega_0$	4·14836	4·22162	3·45521	2·81078	2·26135
		$\omega A/\Omega_0$	5·40776	6·67606	9·95418	—	—
	$\frac{3}{4}$	$\omega R/\Omega_0$	8·17001	6·89216	5·83108	4·93580	4·17047
		$\omega A/\Omega_0$	5·96387	7·49968	11·67621	—	—
	1	$\omega R/\Omega_0$	10·89778	9·44451	8·21087	7·15088	6·23038
		$\omega A/\Omega_0$	6·35313	7·74051	11·41778	—	—

	p \ μ		0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Omega_1 \geqslant a_1$	0	$\omega R/\Omega_0$	—	—	—	1·50164	1·02154
		$\omega A/\Omega_0$	—	2·15691	1·83268	1·62311	—
	$\frac{1}{4}$	$\omega R/\Omega_0$	—	—	—	—	2·51929
		$\omega A/\Omega_0$	2·56392	2·88994	3·19146	3·47099	3·73109
	$\frac{1}{2}$	$\omega R/\Omega_0$	—	—	12·18575	6·73412	5·14836
		$\omega A/\Omega_0$	2·54574	3·15699	3·83067	4·57672	5·40776
	$\frac{3}{4}$	$\omega R/\Omega_0$	—	—	17·69761	10·34444	8·17001
		$\omega A/\Omega_0$	2·83283	3·48651	4·22111	5·05306	5·96387
	1	$\omega R/\Omega_0$	—	—	20·74207	13·29483	10·89778
		$\omega A/\Omega_0$	3·28406	3·93214	4·65099	5·45305	6·35313

Dashes (—) represent cases in which smooth transition does not take place.

solution takes place in the radial direction but not in the azimuthal direction for lower values of λ. It is because the axial flow dominates the circumferential flow. Again in Table 7 ($\Omega_1 \geqslant a_1$) there is a smooth transition in the azimuthal direction whereas in the radial direction, this is not the case. The reason is that the secondary circumferential flow induces a flow along the axis of rotation directed towards the plate. This flow is responsible for creating instability in the radial direction. We, therefore, do not get values of ω_A for smaller values of μ at which low frequency solution agrees with the high frequency solution. For $\Omega_0 = 0$ i.e. the absence of the mean rotational flow, the values of ω_R and ω_A are independent of either λ or μ.

When Ω_1 is very small as compared to a_1, this procedure of obtaining the transition from the low to high frequency solution cannot be adopted in the circumferential direction.

Similarly when a_1 is very small as compared to Ω_1, we do not get smooth transition in the radial direction. The high frequency solution given by (48) and (49) also indicates that for the two extreme cases $\Omega_1 << a_1$ and $a_1 << \Omega_1$ smooth transition cannot occur. This approximation gives fairly good results for transition when λ and μ are comparable to each other.

In both the parts of Table 8, we observe that when Ω_0 the mean circumferential flow, dominates the axial flow, there is almost no smooth transition specially when $a_0 \approx 0$ or $p \approx 0$. It is again due to the instability caused by the rotation in the flow field. The numerical results given in the first part show that the low and high frequency solutions do not overlap for smaller values of λ in the azimuthal direction. In this case the unsteady part of the rotational velocity is small as compared to that of the axial velocity. When $\Omega_0 \leqslant a_0$ and $\Omega_1 \leqslant a_1$ rotational flow dominates the axial flow, we get smooth transition in the azimuthal direction but there is no transition in the radial direction for lower values of μ due to the rotation of the disk which acts as centrifugal fan (see second part of Table 8).

In general whenever the mean or the secondary circumferential flow dominates the mean or the secondary axial flow, it creates an instability in the axial direction. This will be responsible for not giving good results for the values of the frequency at which smooth transition may take place.

FEW PARTICULAR CASES

(*i*) $q = 0$, $\lambda = 0$ give $\Omega_0 = \Omega_1 = 0$. This case can be interpreted as "unsteady flow in the neighbourhood of an axi-symmetric stagnation point". Srivastava[19] who studied this flow has obtained the value $\omega_R = 8 \cdot 0846\ a_0$ at which the transition of one type of flow to the other occurs. Whereas from our solution, the value of this frequency is $7 \cdot 88455 a_0$.

(*ii*) When the mean flow is absent (i.e., $a_0 = 0, \Omega_0 = 0$,) the low frequency solution does not exist. The high frequency solution is valid for all values of the frequency. This case can be interpreted as "torsional oscillations of a disk in a pulsating stream". In the absence of the pulsating stream i.e., $a_1 = 0$, we get the problem of the torsional oscillations of a disk in a fluid at rest. Our solution in the later case is the first order approximation of that obtained by Rosenblat[23]. When $\Omega_1 = 0$, we get the flow of the pulsating stream past a fixed lamina.

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