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Isothermal Shock Wave in Magnetogasdynamics

B. G. VERMA, J. P. VISHWAKARMA & VISHNU SHARAN

Department of Mathematics, University of Gorakhpur
Gorakhpur-273001

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Abstract. The problem of propagation of a plane isothermal discontinuity (shock) wave in a homogeneous semi-infinite body of a perfect gas, in the presence of a magnetic field have been solved. It has been shown that under certain definite conditions, the density ρ at the wave front may be arbitrarily high for a single compression pulse. A certain class of solutions of the present problem for a non-homogeneous semi-infinite body have been derived. Such solutions are expected to be of great importance in compression problems of plasma.

1. Introduction

Kaliski[1] has solved the problem of propagation of a plane isothermal discontinuity wave under an external pressure, and has shown that density behind the wave front depends on the external pressure. He has also obtained a closed form solution[2] for the propagation problem of an isothermal shock wave in a non-homogeneous semi-infinite body.

The solution to the present problem is contained in two sections. In Section 'A', we have formulated the problem and have studied the propagation of a plane isothermal magnetogasdynamic shock wave in a homogeneous semi-infinite body acted on by a prescribed pressure at the boundary. It has been shown that the density behind the wave front depends on the pressure at the boundary of the gas and hence it may be higher, by an arbitrary value than the density ahead of the front. Also, the wave front velocity and the particle velocity behind the wave front are shown to be higher in magnetogasdynamics in comparison to the ordinary case[1]. In Section 'B', a certain class of solutions for the present problem is constructed, for a non-homogeneous body and as an example, a closed solution is obtained for a particular type of non-homogeneity.

This study is expected to be of importance in the problems of plasma compression for the purposes of thermonuclear microfusion.

Section 'A'

2. Problem and Conditions at the Discontinuity Front

We consider a perfect gas with infinite electrical conductivity, under isothermal conditions so that the equations of motion of the gas behind a plane discontinuity wave in presence of an ambient magnetic field, have the form.

$$\frac{\partial (\log \rho)}{\partial t} + u \frac{\partial (\log \rho)}{\partial x} + \frac{\partial u}{\partial x} = 0,$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + a^2 \frac{\partial (\log \rho)}{\partial x} + \frac{h}{\rho} \frac{\partial h}{\partial x} = 0. \tag{1}$$

and

$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} = 0,$$

where symbols have their usual meanings.
The equation of state is

$$p = RT\rho = a^2\rho. \tag{2}$$

Eliminating log ρ, between Eqns. (1) and (2), we have

$$\left(a^2 + V_A^2 - u^2\right) u_{xx} - \ddot{u} - 2\left(uu_x^2 + \dot{u}u_x + u\dot{u}_x\right)$$

$$- \frac{(\dot{u} + uu_x)\, u_x V_A^2}{a^2 + V_A^2} = 0\,, \tag{3}$$

where

$$\left(a^2 + V_A^2\right) \psi_x = -\left(\dot{u} + uu_x\right).$$

$$V_A^2 = \frac{h^2}{\rho}\,, \tag{4}$$

and dot '.' above a quantity stands for the derivative w.r.t. time. We have also substituted log $\rho = \psi$ for the sake of convenience.

The conditions at the discontinuity wave front, the gas ahead of which is at rest, are[1]

$$\rho_2 (D - u_2) = D_1 \rho ,$$

$$h_2 (D - u_2) = D h_1 , \tag{5}$$

$$D u_2 \rho_1 = (\rho_2 - \rho_1) \left\{ a^2 + \frac{k^2}{2} (\rho_2 + \rho_1) \right\},$$

$$\frac{D u_2^2}{2} - \left\{ a^2 + \frac{k^2}{2} (\rho_2 + \rho_1) \right\} \frac{u_2 \rho_2}{\rho_1} + \frac{Q}{\rho_1} + D k^2 (\rho_2 - \rho_1) = 0 , \tag{6}$$

where quantities with suffixes '1' and '2' denote the values of those quantities just ahead and just behind the front and $Q = (\alpha_2 - \alpha_1) \dfrac{\partial T}{\partial x}$, α being the thermal conductivity. The x-axis is taken along the outward drawn normal to the discontinuity wave front, which moves with velocity D. Also, $\dfrac{\partial T}{\partial x} = 0$, $\alpha_2 - \alpha_1 \rightarrow \infty$, and $\dfrac{h}{\rho} = k$ (a constant). When $Q = 0$,

$$D k^2 (\rho_2 - \rho_1) + u_2 \left[\frac{D u_2}{2} - \left\{ a^2 + \frac{k^2}{2} (\rho_2 + \rho_1) \right\} \frac{\rho_2}{\rho_1} \right] = 0 ,$$

so that

$$\frac{D u_2}{2} - \left\{ a^2 + \frac{k^2}{2} (\rho_2 + \rho_1) \right\} \frac{\rho_2}{\rho_1} = 0 ,$$

and

$$(\rho_2 - \rho_1) = 0 ,$$

the latter being the condition that there is no discontinuity.

3. Plane Discontinuity Wave

We assume that a constant pressure p_e given by

$$p_e = a^2 \rho_e , \tag{7}$$

is prescribed at the surface of a semi-infinite body. It is also assumed that the solution is constant between the discontinuity wave and the surface of the semi-infinite body. The Eqn. (3) is then satisfied identically and from the relations (5) at the discontinuity wave front, we have

$$D = \left\{ a^2 + \frac{k^2}{2} (\rho_2 + \rho_1) \right\}^{\frac{1}{2}} \sqrt{\frac{\rho_2}{\rho_1}} , \tag{8}$$

$$u_2 = \left(\sqrt{\frac{\rho_2}{\rho_1}} - \sqrt{\frac{\rho_1}{\rho_2}}\right)\left\{a^2 + \frac{k^2}{2}\ (\rho_2 + \rho_1)\right\}^{\frac{1}{2}} \quad . \tag{9}$$

Also, from the condition (7)

$$\rho_2 = \rho_e, \tag{10}$$

Hence, finally

$$\rho_2 = \rho_e\ ;\ D = \sqrt{\left\{a^2 + \frac{k^2}{2}\ (\rho_e + \rho_1)\right\}\frac{\rho_e}{\rho_1}}\ ,$$

and

$$u_2 = \left(\sqrt{\frac{\rho_e}{\rho_1}} - \sqrt{\frac{\rho_1}{\rho_e}}\right)\left\{a^2 + \frac{k^2}{2}\ (\rho_e + \rho_1)\right\}^{\frac{1}{2}} . \tag{11}$$

The solution has thus been obtained. Since it depends on the boundary condition, ρ_2 may be higher, by an arbitrary value, than ρ_1, while D and u_2 may increase to become arbitrary high. As the terms containing k are present on account of the magnetic field, D and u_2 are higher in magnetogasdynamics than their values in the absence of magnetic field.

The density of energy released at the wave front is

$$Q = \left[(\rho_2 - \rho_1)\left\{a^2 + \frac{k^2}{2}(\rho_1 - \rho_2)\right\}\right]\sqrt{\left\{a^2 + \frac{k^2}{2}\ (\rho_2 + \rho_1)\right\}\frac{\rho_1}{\rho_2}}\ , \tag{12}$$

Section 'B'

4. Solution of Equations for a Certain Class of Cases

We assume that the undisturbed density ρ_1 is variable and has the form

$$\rho_1 = \rho_0\, \phi(x)\,, \tag{13}$$

where

$$\rho_1 = \rho_0 \text{ at } x = 0\ .$$

We shall seek the solution in the following cases :

$$\rho = \rho(t) = \rho_p\, f(t), \tag{14}$$

where $\rho = \rho_p$ at $t = 0$.

Then the Eqn. (1) becomes

$$\frac{\partial\ (\log u)}{\partial t} = -\ \frac{\partial u}{\partial x}\ ,$$

$$\frac{\partial (\log \rho)}{\partial t} = - \frac{\partial u}{\partial x}, \tag{15}$$

$$\frac{\partial (\log h)}{\partial t} = - \frac{\partial u}{\partial x},$$

so that, we have

$$\rho(t)\, F(x) = u. \tag{16}$$

From Eqn. (15), it follows that

$$u = - \frac{\partial (\log \rho)}{\partial t}\, x + B(t) = (t)\, F(x). \tag{17}$$

This equation is satisfied if

$$F(x) = H - Cx,\ H\rho = B(t) \text{ and } C\rho = \frac{\partial (\log \rho)}{\partial t}, \tag{18}$$

where H and C are arbitrary constants. Using Eqn. (14), we obtain from the last equation

$$\rho = \frac{1}{M - Ct}, \tag{19}$$

where $M = \dfrac{1}{\rho_p}$, that is, the solution in ρ must be hyperbolic in x_f.

Now

$$(H - Cx_f)\, \rho(t) = (H - Cx_f)\, \frac{\rho(t)}{\rho_1(x_f)}\, \rho_1(x_f)$$

$$= \left(\sqrt{\frac{\rho_2}{\rho_1}} - \sqrt{\frac{\rho_1}{\rho_2}} \right) \sqrt{\left\{ a^2 + \frac{k^2}{2}(\rho_2 + \rho_1) \right\}}, \tag{20}$$

where Eqns. (9) and (16) have been used, $x = x_f(t)$ being the shock position. Alternatively

$$(H - Cx_f)\, \rho_1(x_f)\, Z^2 = \left\{ a^2 + \frac{k^2}{2}(\rho_2 + \rho_1) \right\}^{\frac{1}{2}} \left(Z - \frac{1}{Z} \right), \tag{21}$$

where $Z = \rho_2/\rho_1$.

The cases to be considered are those of

$$Z \gg 1. \tag{22}$$

Then, from Eqn. (21), we have

$$Z = \frac{\left(a^2 + \dfrac{k^2}{2}\ \rho_1 \right)^{\frac{1}{2}}}{\left\{ (H - Cx_f)^2 \rho_1^2 - \dfrac{k^2}{2}\ \rho_1 \right\}^{\frac{1}{2}}}, \tag{23}$$

$$u_2 = \frac{(H - Cx_f)\left\{a^2 + \frac{k^2}{2}\ \rho_1\right\}}{(H - Cx_f)^2\rho_1 - \frac{k^2}{2}}, \tag{24}$$

and

$$\rho_2 = \frac{\left(a^2 + \frac{k^2}{2}\rho_1\right)}{(H - Cx_f)^2\rho_1 - \frac{k^2}{2}}. \tag{25}$$

At the discontinuity wave front $x_f(t)$, we have

$$x_f = \int_0^t D(t)\, dt, \tag{26}$$

which for constant D yields,

$$x_f = D\,(t). \tag{27}$$

Also, using Eqns. (13) and (25) in Eqn. (8), we have

$$D = \frac{\left(a^2 + \frac{k^2}{2}\ \rho_1\right)(H - Cx_f)}{\rho_0(H - Cx_f)\,\phi(x) - \frac{k^2}{2}}, \tag{28}$$

which gives for its solution

$$x_f = \psi(D).$$

Hence

$$D = \psi'(D).\ D',$$

which on integration, gives

$$t + N = \int \frac{dD}{D}\ \psi'(D). \tag{29}$$

This solution must be in agreement for ρ given by Eqn. (19). This can be achieved easily by introducing into the Eqn. (8), ρ in its explicit from Eqn. (19) and ρ_1 given by the Eqn. (13).

Now, as an example, we consider a case in which a closed form solution is obtained.

A closed-form solution : We assume that the density $\rho_1(x)$ is of the form

$$\rho_1(x) = \frac{a^2(H - Cx) + \frac{Dk^2}{2}}{(H - Cx)\left\{D(H - Cx) - \frac{k^2}{2}\right\}}, \tag{30}$$

where for $x = 0$, $\rho_1(0) = \rho_0$, and shock velocity D is taken to be constant. Hence

$$H = \frac{1}{2D}\left[\left(\frac{k^2}{2} + \frac{a^2}{\rho_0}\right) + \sqrt{\left(\frac{k^2}{2} + \frac{a^2}{\rho_0}\right)^2 + \frac{2D^2k^2}{\rho_0}}\,\right]. \tag{31}$$

On substituting Eqn. (30) into Eqn. (25) and making use of Eqn. (8), we have

$$D = \sqrt{\left\{a^2 + \frac{k^2}{2}(\rho_2 + \rho_1)\right\}\frac{\rho_2}{\rho_1}} \tag{32}$$

which becomes an identity. That is, the above form of undisturbed density in Eqn. (30) agrees with the solution (29). From Eqn. (32), we obtain

$$\frac{\rho_2}{\rho_1} = \frac{2D^2}{\left(a^2 + \frac{k^2}{2}\rho_1\right) + \sqrt{\left(a^2 + \frac{k^2}{2}\rho_1\right)^2 + 2D^2k^2\rho_1}}. \tag{33}$$

In order to satisfy the condition (22), it is assumed that

$$\frac{2D^2}{\left(a^2 + \frac{k^2}{2}\rho_1\right) + \sqrt{\left(a^2 + \frac{k^2}{2}\rho_1\right)^2 + 2D^2k^2\rho_1}} \geqslant 1 \tag{34}$$

From Eqns. (30), (25) and (27), we find

$$\rho_2 = \frac{1}{\frac{H}{D} - Ct}. \tag{35}$$

Hence. from Eqn. (19), we have

$$M = \frac{H}{D} = \frac{1}{2D^2}\left[\left(\frac{k^2}{2} + \frac{a^2}{\rho_0}\right) + \sqrt{\left(\frac{k^2}{2} + \frac{a^2}{\rho_0}\right)^2 + \frac{2D^2k^2}{\rho_0}}\,\right], \tag{36}$$

and

$$D^2 = \rho_p\left(\frac{k^2}{2} + \frac{a^2}{\rho_0}\right) + \frac{k^2\rho_p^2}{2\rho_0}. \tag{37}$$

The only condition to be satisfied is that

$$\frac{H}{D} = \frac{1}{2D^2}\left[\left(\frac{k^2}{2} + \frac{a^2}{\rho_0}\right) + \sqrt{\left(\frac{k^2}{2} + \frac{a^2}{\rho_0}\right)^2 + \frac{2D^2k^2}{\rho_0}}\,\right] > C\tau, \tag{38}$$

where τ is the finite time interval in which the process is studied and C is arbitrary, as already assumed.

The above conditions give the following results. Using $\frac{1}{\rho_p} = M = \frac{H}{D}$ in Eqn. (19), we have found :

$$\rho(t) = \rho_p \frac{1}{1 - \rho_p Ct}, \tag{39}$$

and

$$\rho_1(x_f) = \frac{2a^2 \rho_p(1 - Ct\,\rho_p) + k^2\rho_p^2}{(1 - Ct\,\rho_p)\,\{2D^2(1-Ct\rho_p) - k^2\rho_p\}}. \tag{40}$$

By virtue of Eqn. (17), we have

$$u = \frac{(D - Cx\rho_p)}{(1 - \rho_p\,Ct)}, \tag{41}$$

and

$$u_2 = D, \tag{42}$$

where D is given by the Eqn. (37).

The last equation is a consequence of the strong shock condition (22). The Eqns. (39), (41) and (42) thus constitute the solution of our problem.

From Eqns. (37) and (42) it is concluded that the shock velocity and consequently the particle velocity are increased due to the presence cf the magnetic field.

References

1. Kaliski, S., *Bull. Acad. Polon. Sci., Ser. Sci. Techn.*, **24** (1976), 463.
2. Kaliski, S., *Bull. Acad. Polon. Sci., Ser. Sci. Techn.*, **24** (1976), 467.