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Flow of an Incompressible Second-Order Fluid Past a Sphere

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Abstract. The steady axisymmetric flow of an incompressible second-order fluid past a sphere at rest is considered by the method of Blasius with a potential flow in the main stream. The first four terms of the series are obtained by Meksyn's method. The position of the separation ring is calculated for various values of the second-order parameters. The position of the separation ring for the Newtonian case agrees very nearly with that obtained by Schlichting who used exact values of the first four terms of the series. The effect of second-order parameters on the position of the separation ring is to advance it towards the forward stagnation point.

1. Introduction

The solutions of problems of engineering interest in the flow of visco-elastic fluids require a good understanding of the behaviour of such fluids under a variety of circumstances. The present paper is concerned with the steady boundary layer flow of an incompressible second-order fluid past a sphere. An incompressible second-order fluid proposed by Coleman & Noll[1] describes qualitatively correct behaviour of many fluids under retarded motions. The rectangular cartesian components of the Cauchy stress τ_{ij} and the fluid motion, in such a fluid, are assumed to be related as follows :

$$\tau_{ij} = - \tilde{p}\, \delta_{ij} + \mu_1 A_{(1)ij} + \mu_2 A_{(2)ij} + \mu_3 A_{(1)i\alpha} \overset{*}{A}_{(1)\,j} \tag{1}$$

where

$$A_{(1)ij} = v_{i,j} + v_{j,i}$$

and

$$A_{(2)ij} = a_{i,j} + a_{j,i} + 2v_{m,i}\, v^{m}, j \tag{2}$$

The v_i and a_i are the components of the velocity and acceleration vectors respectively. The $\tilde{p}$ is a constitutively indeterminate pressure which differs, in general, from the mean pressure $p = -\frac{1}{3}(\tau_{11} + \tau_{22} + \tau_{33})$, and μ_1, μ_2, μ_3 are material constants. The case $\mu_2 = \mu_3 = 0$ corresponds to an incompressible Newtonian fluid. On thermodynamic considerations μ_2 is found to be negative. These material constants have been determined experimentally for solutions of poly-isobutylene in cetane of various concentrations by Markovitz & Brown.

The above flow problem is solved by the method of expansion of flow functions in series and assuming the potential flow in the main stream. The first four terms of the series are obtained by the method used by Meksyn[3]. The position of the separation ring on the sphere is calculated for various values of the second-order parameters. For the case of Newtonian fluid, the position of the separation ring is found to be at 109.2°. Schlichting[4] calculated the position of the separation ring on the sphere for the Newtonian fluid by the method of series expansion and on the assumption of potential flow in the main stream. Using the numerically exact values of the first four terms of the series, he calculated the position of the separation ring at 109.6°. The corresponding value obtained by us is in good agreement with this value. Hence, we can assume that the other values for different second-order parameters are fairly correct.

2. Boundary Layer Equations

Consider a steady axisymmetric flow of an incompressible second-order fluid with velocity U_∞ at infinity past a sphere of radius a. Assume that the velocity of the fluid in the main stream is given by

$$U = U_\infty \sin\theta \tag{3}$$

where θ is the angle at the centre of the sphere measured from the axis of symmetry. Let x and z denote the distances measured along and perpendicular respectively to the surface of the sphere, so that $x = a\theta$. Let u and w denote the velocity components along x and z respectively. The equations of continuity and motion are given by

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} + \frac{1}{a}\, u \cot\left(\frac{x}{a}\right) = 0 \tag{4}$$

$$u\,\frac{\partial u}{\partial x} + w\,\frac{\partial u}{\partial z} = U\,\frac{\partial U}{\partial x} + \nu_1\,\frac{\partial^2 u}{\partial z^2} - \frac{\nu_2}{R}\,\frac{\partial R}{\partial x}\left\{\left(\frac{\partial u}{\partial z}\right)^2 + 2u\,\frac{\partial^2 u}{\partial z^2}\right\} - \nu_3\left\{\frac{1}{R}\,\frac{\partial R}{\partial x}\left(\frac{\partial U}{\partial z}\right)^2 + \frac{1}{2}\,\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right)^2 - w\frac{\partial^3 u}{\partial z^3} + \frac{2}{R}\,\frac{\partial R}{\partial x}\left(u\,\frac{\partial^2 u}{\partial z^2}\right) - \frac{\partial}{\partial x}\left(u\,\frac{\partial^2 u}{\partial z^2}\right)\right\} \tag{5}$$

where $\nu_i = \frac{1}{\rho}\mu_i$, $i = 1, 2, 3$ and ρ is the density of the fluid. $R = a \sin\theta$.

The boundary conditions of the problem are

$$u = 0, \quad w = 0 \quad \text{at} \quad z = 0$$

$$u \to U(x) \quad \text{as} \quad z \to \infty \tag{6}$$

we assume the following expansions for u and w:

$$u = U_\infty \theta \left(f_1' - \frac{1}{3} f_3' \theta^2 + \frac{1}{40} f_5' \theta^4 - \frac{1}{1260} f_7' \theta^6 + \ldots \right)$$

$$w = \left(\frac{U_\infty \nu_1}{2a}\right)^{1/2} \left[-2 f_1 + \left(\frac{4}{3} f_3 + \frac{1}{3} f_1 \right) \theta^2 - \left(\frac{3}{20} f_5 + \frac{1}{9} f_3 - \frac{1}{45} f_1 \right) \theta^4 + \left(\frac{2}{315} f_7 + \frac{1}{120} f_5 - \frac{1}{135} f_3 + \frac{2}{945} f_1 \right) \theta^6 + \ldots \right] \tag{7}$$

where a dash denotes derivative w. r. t. η which is defined by

$$\eta = \left(\frac{2U_\infty}{\nu_1 a}\right)^{1/2} z.$$

The form (7) of u and w satisfy the Eqn. (4) of continuity. The boundary conditions on f_1, f_3, f_5, f_7 etc. are

$$f_i = 0, \quad f_i' = 0, \quad i = 1, 3, 5, 7, \quad \text{at} \quad \eta = 0$$

$$f_1' = 1, \; f_3' = \frac{1}{2}, \; f_5' = \frac{1}{3}, \; f_7' = \frac{1}{4} \quad \text{at} \quad \eta \to \infty \tag{8}$$

Substituting the expansions of u, w and U in the Eqn. (5) and equating the coefficients of like powers of θ to zero, we obtain the following set of ordinary differential equations for f_1, f_3, f_5 and f_7

$$f_1''' + f_1 f_1'' = -\frac{1}{2} + \frac{1}{2} f_1'^2 + \alpha\,(f_1''^2 + 2 f_1' f_1''') + 2\beta\,(f_1''^2 + f_1 f_1^{iv}) \tag{9}$$

$$f_3''' + f_1 f_3'' = -1 - \frac{1}{2} f_1 f_1'' + 2\,(f_1' f_3' - f_1'' f_3) + 2\beta \left(\frac{1}{2} f_1''^2 + \frac{1}{2} f_1 f_1^{iv} + f_1' f_1''' \right) + f_1 \left(f_3^{iv} - f_1' f_3''' + 3 f_1'' f_3'' - f_1''' f_3' + 2 f_1^{iv} f_3 \right) + 2\alpha \left(\frac{1}{2} f_1''^2 + f_1' f_1''' + f_1' f_3''' + f_1''' f_3'' + f_1''' f_3' \right) \tag{10}$$

$$
\begin{aligned}
f_5''' + f_1 f_5'' = & -\frac{8}{3} + \frac{4}{9} f_1 f_1'' - \frac{20}{9}(f_1 f_3'' + f_1'' f_3) + \frac{20}{9}(3 f_3'^2 \\
& - 4 f_3 f_3'') + 3(f_1' f_5' - f_1'' f_5) - \frac{8}{9}(\alpha+\beta)(f_1''^2 \\
& + 2 f_1' f_1''') + \frac{80}{9}(\alpha+\beta)(f_1' f_3''' + f_1'' f_3'' + f_1''' f_3') \\
& + \frac{8\beta}{9}(-f_1 f_1^{iv} + 5 f_1^{iv} f_3 + 5 f_1 f_3^{iv}) + 2\beta\Big(\frac{80}{9} f_3''^2 \\
& + \frac{80}{9} f_3 f_3^{iv} - \frac{80}{9} f_3' f_3''' + f_1 f_5^{iv} - 2 f_1' f_5''' + 4 f_1'' f_5'' \\
& - 2 f_1''' f_5' + 3 f_1^{iv} f_5\Big) + 2\alpha\Big(\frac{20}{9} f_3''' + \frac{40}{9} f_3' f_3'' \\
& + f_1' f_5''' + f_1'' f_5'' + f_1''' f_5'\Big)
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
f_7''' + f_1 f_7'' = & -8 - \frac{4}{3} f_1 f_1'' + \frac{14}{3}(f_1'' f_3 + f_1 f_3'') - \frac{70}{3} f_3 f_3'' \\
& - \frac{21}{4}(f_1'' f_5 + f_1 f_5'') + \frac{21}{2}(4 f_3' f_5' - 3 f_3'' f_5 - 2 f_3 f_5'') \\
& + 4(f_1' f_7' - f_1'' f_7) + 2(\alpha+\beta)\{(\frac{4}{3} f_1''^2 + 2 f_1' f_1''') \\
& - \frac{28}{3}(f_1'' f_3'' + f_1' f_3''' + f_1''' f_3') + \frac{70}{3}(f_3''^2 + 2 f_3' f_3''') \\
& + \frac{21}{2}(f_1'' f_5'' + f_1' f_5''' + f_1''' f_5')\} + 2(\alpha+5\beta)\Big\{\frac{21}{2}(f_3'' f_5'' \\
& + f_3' f_5''' + f_3''' f_5) + (f_1'' f_7'' + f_1' f_7''' + f_1''' f_7')\Big\} \\
& + 2\beta\Big(\frac{4}{3} f_1 f_1^{iv} - \frac{14}{3} f_1 f_3^{iv} + \frac{70}{3} f_3 f_3^{iv} + \frac{21}{4} f_1^{iv} f_5 \\
& + \frac{21}{4} f_1 f_5^{iv} + \frac{63}{2} f_3^{iv} f_5 + 21 f_3 f_5^{iv} - \frac{14}{3} f_1^{iv} f_3 \\
& + 4 f_1^{iv} f_7 + f_1 f_7^{iv}\Big)
\end{aligned}
\tag{12}
$$

where, $\alpha = \frac{\nu_2 U_\infty}{a \nu_1}$ and $\beta = \frac{\nu_3 U_\infty}{a \nu_1}$.

3. Solutions of Equations

The Eqn. (9), subject to the boundary conditions (8) has been solved by Saroa[5] by the method of series expansion followed by Laplace's method. This method has given sufficiently accurate results. We use this method to solve the remaining three equations.

We express the functions $f_i(\eta)$ in power series of η,

$$f_i(\eta) = \frac{1}{2!} a_i \eta^2 + \frac{1}{3!} b_i \eta^3 + \frac{1}{4!} c_i \eta^4 + \frac{1}{5!} d_i \eta^5 + \frac{1}{6!} e_i \eta^6 + \dots,$$

where $i = 1, 3, 5, 7, \dots$ (13)

This form of f_i satisfies the boundary conditions (8) at $\eta = 0$. The other constants a_i, b_i, c_i, d_i, e_i etc. are to be determined. The Eqn. (13) are valid only for sufficiently small values of η. Substituting these expansions of f_3, f_5, f_7 in the Eqns. (10) to (12), and equating the co-efficients of different powers of η to zero, we determine the constants b_i, c_i, d_i, e_i etc. in terms of a_i only. The constants a_i are to be determined by using the boundary condition (8) at $\eta \to \infty$. To make use of this conditions, we write the Eqns. (10) to (12) in the form,

$$f_i''' + f_1 f_i'' = H_i(\eta), \quad i = 3, 5, 7 \dots \tag{14}$$

where $H_i(\eta)$, $i = 3, 5, 7$ are the right hand sides of the Eqns. (10) to (12) respectively Letting

$$F(\eta) = \int_0^\eta f_1(\eta)\, d\eta \tag{15}$$

and

$$Q_i(\eta) = a_i + \int_0^\eta e^F H_i(\eta)\, d\eta \tag{16}$$

By integrating twice the Eqn. (14), we get

$$f_i'(\eta) = \int_0^\eta e^{-F} \phi_i(\eta)\, d\eta. \tag{17}$$

This form can be evaluated by Laplace's method. The coefficients a_3, a_5, a_7 are given respectively by

$$\int_0^\infty e^{-F} \phi_3(\eta)\, d\eta = \frac{1}{2} \tag{18}$$

$$\int_0^\infty e^{-F} \phi_5(\eta)\, d\eta = \frac{1}{3} \tag{19}$$

$$\int_0^\infty e^{-F} \phi_7(\eta)\, d\eta = \frac{1}{4} \tag{20}$$

These integrals can be evaluated asymptotically by Laplace's method. Putting $F = \tau$, transforming the Eqns. (18) to (20) to the variable τ and integrating in the gamma functions, we find,

$$k_i = \tfrac{1}{3}\left[P_i \Gamma_{(1/3)} + Q_i \Gamma_{(2/3)} + R_i \Gamma_{(1)} + S_i \Gamma_{(4/3)} + T_i \Gamma_{(5/3)} \right]. \tag{21}$$

where

$$k_3 = \tfrac{1}{2},\ k_5 = \tfrac{1}{3},\ k_7 = \tfrac{1}{4}$$

and

$$P_i = \lambda_i a_i$$

$$Q_i = a_i \lambda_2 + b_i \lambda_1^2$$

$$R_i = \frac{a_i}{a_1}\left(-\frac{3}{10}\frac{c_i}{a_1} + \frac{3}{8}\frac{b_1^2}{a_1^2}\right) - \frac{3}{2}\frac{b_1 b_i}{a_1^2} + 3\frac{c_i}{a_1}$$

$$S_i = a_i \lambda_4 + \frac{1}{6} b_i (8\lambda_1 \lambda_3 + 3\lambda_2^2) + c_i \lambda_1^2 \lambda_2 + \frac{\lambda_1^4}{6}(d_i + a_1 a_i)$$

$$T_i = a_i \lambda_5 + \frac{5}{12} b_i (3\lambda_1 \lambda_4 + 2\lambda_2 \lambda_3) + \frac{5}{2} c_i \left(\frac{\lambda_1^2 \lambda_3}{3} + \frac{\lambda_1 \lambda_2^2}{4}\right)$$

$$+ \frac{5}{12} \lambda_1^3 \lambda_2 (d_i + a_1 a_i) + \frac{\lambda_1^5}{24} (e_i + 4a_1 b_i + b_1 a_i). \tag{22}$$

The Eqn. (21) determine the three unknowns a_3, a_5, a_7. So this Eqn. (21) are in general divergent. We use Euler's transformation,

$$\sum_{n=0}^{\infty} (-1)^n a_n = \sum_{n=0}^{\infty} (-1)^n \frac{\Delta^n a_0}{2^{n+1}} \tag{23}$$

where

$$\Delta a_m = a_{m+1} - a_{m1},\ \Delta^2 a_m = \Delta a_{m+1} - \Delta a_{m1}, \ldots$$

To determine a_3, a_5, a_7 from the Eqn. (21), we took five terms of the series and applied Euler's transformation once starting from third term. The value of a_1 has been taken from Saroa[5]. The value of a_3, a_5, a_7 have been determined for $-\alpha = \beta = 0, 0.03, 0.06$ and the values have shown in Table 1.

Table 1. Values of a_i, $i = 1, 3, 5, 7$ for different values of (α, β)

α, β	0.0	−0.03, 0.03	−0.06, 0.06
a_1	0.9308	0.9525	0.9758
a_3	1.109453	1.193449	1.39972
a_5	1.709728	2.396625	3.401492
a_7	3.973331	8.585847	12.94569

4. Discussion

The shearing stress to on the wall of the sphere is given by

$$\tau_0 = \mu_1 \left[\frac{\partial u}{\partial z}\right]_{z=0}$$

$$= \left(\frac{2U_\infty}{\nu_1 a}\right)^{1/2} \rho\, \nu_1 U_\infty \theta \left[f_1''(0) - \frac{1}{3} f_3''(0)\, \theta^2 \right.$$

$$\left. + \frac{1}{40} f_5''(0)\, \theta^4 - \frac{1}{1260} f_7''(0)\, \theta^6 \right] \tag{24}$$

The position of the separation ring can be obtained by the condition that the shearing stress to the surface must vanish there.

For a Newtonian fluid ($\alpha=\beta=0$), the condition that the shearing stress τ_0 at the surface vanishes is given by

$$0.003153\, X^3 - 0.042743\, X^2 + 0.369818\, X - 0.9308 = 0 \tag{25}$$

where $X = \theta^2$. The acceptable solution of this cubic is $X = 3.632$ which gives $\theta = 109.2°$. Thus for the Newtonian case, the separation occurs at 109.2°. Schlichting calculated this values to be 109.6° by using the exact values $f_1''(0)$, $f_3''(0)$, $f_5''(0)$ and $f_7''(0)$. Thus the method used is fairly correct.

For a second-order fluid, the conditions of the vanishing the shearing stress at the surface for $-\alpha=\beta=0.03$, $-\alpha=\beta=0.06$ respectively are

$$0.006814\, X^3 - 0.059915\, X^2 + 0.3978\, X - 0.9525 = 0 \tag{26}$$

$$0.0102743\, X^3 - 0.085037\, X^2 + 0.466573\, X - 0.9758 = 0 \tag{27}$$

Solving these cubics, the acceptable roots are respectively $X=3.509$, $X=3.271$. That is the position of the separation ring is at $\theta = 107.3°$ and $\theta = 103.6°$.

This shows that the effect of second-order terms in the constitutive equation on the position of the separation ring is to advance it towards the forward stagnation point. The second order effect is exhibited through the non-dimensional parameter $\alpha = \dfrac{U_\infty \mu_2}{a\, \mu_1}$. Thus the point of separation depends on the material constants μ_1 and μ_2 and also the flow parameters U_∞ and a. This is a peculiarity of a second-order fluid since the point of separation for a Newtonian fluid is independent of those quantities.

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