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Self-Similar Flow Behind a Spherical Radiation-Driven Shock Wave in Magnetogasdynamics

J. B. SINGH & R. K. RAM

Department of Mathematics & Physics, S.G.R. Postgraduate College, Jaunpur-222149

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Abstract. Adiabatic and isothermal self-similar flows behind a radiation-driven shock wave in the presence of magnetic field have been investigated by using the generalised method of Laumbach and Probstein for magnetogasdynamics. The density and magnetic field distributions of the gas ahead of the shock have been assumed to be constant.

1. Introduction

Ramsden and Savic[1] have studied the steady, one-dimensional propagation of a radiation supported detonation wave while Raizer[2] has discussed in detail the possible mechanisms that can occur in the course of heating of a gas by lasers. Daiber et al[3] have proposed that a spherical, radiation driven shock wave could be generated by a number of lasers focussed on a common point so that the entire solid angle is uniformly filled with radiation and propagates with three-fifth power of a time. Champetier et al[4] and Wilson and Turcotte[5] have discussed a self-similar flow behind a spherical radiationdriven shock wave and obtained the exact numerical solutions. Ranga Rao and Ramana[6] have investigated an approximate analytical solutions for the self-similar flows behind a radiation-driven shock wave using Laumbach and Probstein method.

In this paper, we have studied the self-similar flow behind a radiation driven shock wave in the presence of azimuthal magnetic field by using the generalised method of Laumbach and Probstein[7] in magnetogasdynamics. In this problem the magnetic field is assumed to be an idealised field such that lines of force lie on a portion of sphere enclosing the origin and directed tangentially to the advancing shock front[8]. The gas is optically thin and thick ahead and behind the shock respectively and its flow is particle isentropic behind the shock wave. Radiation propagating radially inwards with constant power input P, is completely absorbed in the shock layer of spherically expanding shock wave. Reradiation from the shock layer is not considered. The density and magnetic field distributions of the gas are constant ahead of the shock. The geometry of the shock profile can be seen in Summers[8] and Wilson and Turcotte[5].

The energy content of the flow behind the shock increases due to the absorption of radiation as it propagates. So, it can be assumed here following Ranga Rao and Ramana[6] that the total energy content E of the flow behind the shock is dependent on the shock radius R according to the power law

$$E = E_0 R^n \quad (n > 0) \tag{1}$$

where E_0 is a constant.

The basic dimensional constants involved in this problem are ρ_0, $p_0 = h_0^2$, E_0 and P following Verma and Singh[9] and there exist two dimensionally independent parameters X and Y such that

$$[X] = [h_0^2] \tag{2}$$

$$[Y] = \left[\frac{E_0}{\rho_0}\right] = \left[\frac{P}{\rho_0}\right] \tag{3}$$

and hence it discloses that $n = \frac{5}{3}$. Numerical calculations have been made for $\gamma = \frac{5}{3}$ and $M_h = 10$ and illustrated through the figures.

2. Equation for Adiabatic Flow

The method of Laumbach and Probstein[7] in the generalised form has been presented here in brief, the full details may be found for gasdynamics in Ranga Rao and Purohit[10,11] and for magnetogasdynamics in Verma and Vishwakarma[12] and Verma and Singh[13,14]. The equations of continuity, momentum, magnetic field and energy in the integrated form in Lagrangian formulation are

$$\int_0^R r^2\, dr = \int_{r_0}^R \frac{\rho_0}{\rho}\, r_0^2\, dr_0 \tag{4}$$

$$p^*(r_0, t) - p_s^*(R) = \int_{r_0}^R \frac{1}{r^2} \frac{d^2 r}{dt^2}\, \rho_0 r_0^2\, dr_0 + \int_{r_0}^R \frac{h^2}{\rho r^3}\, \rho_0 r_0^2\, dr_0 \tag{5}$$

$$h(r_0, t) - h_s(R) = \int_{r_0}^R \left[\frac{1}{r^2} + \frac{1}{ru^2} \frac{d^2 r}{dt^2}\right] h_0 r_0\, dr_0 \tag{6}$$

$$\frac{p(r_0, t)}{p_s(r_0)} = \left[\frac{\rho(r_0, t)}{\rho_s(r_0)}\right]^\gamma \tag{7}$$

where

$$p^* = p + \frac{h^2}{2}$$

Since the magnetic field is 'Frozen in Field', we have

$$h\, r\, dr = h_0\, r_0\, dr_0 \text{ (cylindrical form).} \tag{8}$$

The Eqn. (5) with the help of Eqn. (8) can be written as

$$p^*(r_0, t) - p_s^*(R) = \int_{r_0}^{R} \frac{1}{r^2}\left(\frac{\partial^2 r}{\partial t^2}\right)\rho_0\, r_0^2\, dr_0 + \int_{r_0}^{R} \frac{1}{r^3}\left(\frac{\partial r}{\partial r_0}\right)^{-1} h_0^2\, r_0^2\, dr_0 \tag{9}$$

where r_0, r, t and R denote the position of a fluid particle at $t = 0$, the point of explosion, the Eulerian coordinate of a fluid particle of thickness dr, the time and the shock radius respectively.

Now we consider energy conservation equation in the integral form

$$\frac{E}{4\pi} = \frac{E_0\, R^{5/3}}{4\pi} = \int_0^R \frac{p}{\gamma-1}\, r^2\, dr + \int_0^R \frac{h^2}{2}\, r^2\, dr + \int_0^R \frac{1}{2}\left(\frac{\partial r}{\partial t}\right)^2 \rho_0\, r_0^2\, dr_0. \tag{10}$$

This Eqn. (10) with the help of Eqn. (8) yields

$$\frac{E}{4\pi} = \frac{E_0\, R^{5/3}}{4\pi} = \int_0^R \frac{p^*}{\gamma-1}\, r^2\, dr + \frac{\gamma-2}{2(\gamma-1)}\int_0^R \left(\frac{\partial r}{\partial r_0}\right)^{-1} h_0^2\, r_0^2\, dr_0$$

$$+ \frac{1}{2}\int_0^R \left(\frac{\partial r}{\partial t}\right)^2 \rho_0\, r_0^2\, dr_0. \tag{11}$$

For a strong shock, boundary conditions at the shock front are given below

$$\rho_S = \frac{1}{\beta}\,\rho_0 \tag{12}$$

$$p_S = (1-\beta)\,\rho_0\,\dot{R}^2 \tag{13}$$

$$h_s = \frac{1}{\beta}\,h_0 \tag{14}$$

and

$$u_s = (1-\beta)\,\dot{R} \tag{15}$$

where the subscripts S refers to the value at the shock front and R is the location of shock front and a dot above it denotes differentiation with respect to time.

The integral in Eqn. (9) is evaluated approximately by replacing the integrands

$$\frac{1}{r^2}\left(\frac{\partial^2 r}{\partial t^2}\right) \text{ and } \frac{1}{r^3}\left(\frac{\partial r}{\partial r_0}\right)^{-1}$$

by their values at the shock (appendix I).

Then using Eqns. (13), (14) and (6), we get

$$p(r_0, t) = (1-\beta)\rho_0 \dot{R}^2 + \frac{h_0^2}{2\beta^2} + \frac{\rho_0 R}{3}\left(\frac{\partial^2 r}{\partial t^2}\right)_s (1-\xi^3) + \frac{h_0^2}{3}\left(\frac{\partial r}{\partial r_0}\right)_s^{-1}$$
$$\times (1-\xi^3) - \frac{1}{2}\left[\frac{h_0}{\beta} + \frac{h_0}{2}\left\{1+R\left(\frac{\partial r}{\partial t}\right)_s^{-2}\left(\frac{\partial^2 r}{\partial t^2}\right)_s\right\}(1-\xi^2)\right]^2. \tag{16}$$

or

$$p^*(r_0, t) = (1-\beta)\rho_0 \dot{R}^2 + \frac{h_0^2}{2\beta^2} + \frac{\rho_0 R}{3}\left(\frac{\partial^2 r}{\partial t^2}\right)_s (1-\xi^3)$$
$$+\frac{h_0^2}{3}\left(\frac{\partial r}{\partial r_0}\right)_s^{-1}(1-\xi^3).$$

Replacing the integrand $p^*(r_0, t)$ by $p^*(0, t)$ which can be found out from Eqn. (16),

$$\left(\frac{\partial r}{\partial t}\right) \text{ by } \left(\frac{\partial r}{\partial t}\right)_s \text{ and } \left(\frac{\partial r}{\partial r_0}\right) \text{ by } \left(\frac{\partial r}{\partial r_0}\right)_s$$

the Eqn. (11), after integration, reduces to a second order differential equation in R, which after a couple of integration yields

$$R = K t^{\delta} \left(\delta = \frac{3}{5}\right) \tag{17}$$

where K is a constant and it includes other constants which occur in the duration of integration. This constant of proportionality are insignificant while finding the solutions in the form $\frac{u}{u_s}$, $\frac{\rho}{\rho_s}$, $\frac{h}{h_s}$ and $\frac{p}{p_s}$ as a function of $\lambda = \frac{r}{R}$. Using Eqn. (17) we get from Eqn. (16) and (7)

$$\frac{p(r_0, t)}{p_s(R)} = 1+D(1-\xi^3) + \frac{1}{A}\left[1-\{1+G(1-\xi^2)\}^2\right] \tag{18}$$

$$\frac{\rho(r_0, t)}{\rho_s(R)} = 1+D(1-\xi^3) + \frac{1}{A}\left[1-\{1+G(1-\xi^2)\}^2\right]^{1/\gamma}\xi^{a} \tag{19}$$

$$\frac{h(r_0, t)}{h_s(R)} = 1+G(1-\xi^2) \tag{20}$$

where

$$\xi = \frac{r_0}{R},\ a = \frac{2(1-\delta)}{\gamma\delta},\ A = 2\,\beta^2(1-\beta)\,M_h^2$$

$$D = \frac{1}{3}\left[1 + \frac{3M_h^2\,\beta^3}{1 + M_h^2\,\beta^2(\gamma - \gamma\beta - \beta)}\right]\frac{(\delta-1)}{\delta}\,[1+2\,\gamma\,M_h^2\,\beta^2(1-\beta)^2]$$

$$\times\frac{\beta}{3(1-\beta)\{1 + M_h^2\,\beta^2(\gamma - \gamma\beta - \beta)\}} + \frac{1}{3\,\beta(1-\beta)\,M_h^2}$$

$$G = \frac{\beta}{2}\left[1 + \frac{1}{1-\beta}\left\{1 + \frac{3M_h^2\,\beta^3}{1 + M_h^2\,\beta^2(\gamma - \gamma\beta - \beta)}\right\}\right]\frac{\delta-1}{\delta}$$

$$+\frac{\beta}{(1-\beta)^2\{1+M_h^2\,\beta^2(\gamma-\gamma\beta-\beta)}\,[1+2\,\gamma\,M_h^2\,\beta^2(1-\beta)^2]$$

and $M_h = \dfrac{R\sqrt{\rho_0}}{h_0}$ is known as Alfven Mach number.

Making use of Eqn. (4), we get

$$\lambda^3 = 3\,\beta\int_0^{\xi}\frac{\xi^b\,d\xi}{[1+D(1-\xi^3) + \frac{1}{A}\{1 - <1+G(1-\xi^2)>^2\}]^{(1/\gamma)}} \tag{21}$$

with $b = \dfrac{2(\gamma\delta - 1 + \delta)}{\gamma\delta}$, which relates the reduced Eulerian $\lambda = \dfrac{r}{R}$ with reduced Lagrangian $\xi = \dfrac{r_0}{R}$. Since at the shock both λ and ξ take the value 1, the Eqn. (4) takes the form (Laumbach and Probstein[15], on page 837)

$$\frac{1}{\beta} = 3\int_0^1 \frac{\rho_s}{\rho_0}\,\xi^2\,d\xi. \tag{22}$$

The unknown parameter β can be found out by iterating Eqn. (22).

The expression for velocity is obtained by differentiating Eqn. (21) with respect to time t and using Eqn. (15) as

$$\frac{u}{u_s} = \left[\frac{\beta}{(1-\beta)\,\lambda^2}\right]$$

$$\times\int_0^{\xi}\frac{\left[a(1+D) - D\left(a + \frac{3}{\gamma}\right)\xi^3\right] + \frac{1}{A}\left[a\{1 - <1+G(1-\xi^2)>^2\} + \frac{4G\,\xi^2}{\gamma}<1+G(1-\xi^2)>\right]\xi^b}{\left[1+D(1-\xi^3) - \frac{1}{A}\{1 - <1+G(1-\xi^2)>^2\}\right]^{\frac{\gamma+1}{\gamma}}}\,d\xi \tag{23}$$

The Eqn. (21) helps us to calculate the values of $\frac{u}{u_s}$, $\frac{\rho}{\rho_s}$, $\frac{p}{p_s}$ and $\frac{h}{h_s}$ as function of λ.

The flow variables are presented in Figs. 1 to 4 as smooth curves, while the dotted curves denote the analytical solutions of Ranga Rao and Ramana[6].

3. Equations for Isothermal Flow

The flow behind the shock is rendered approximately isothermal when the flows take place at very high temperature, the temperature gradient being zero throughout the flows.

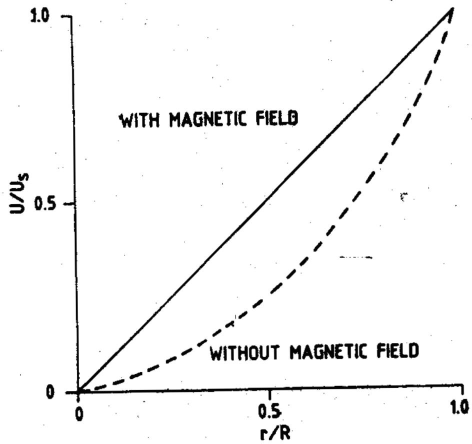


Figure 1. Adiabatic flow; velocity distribution.

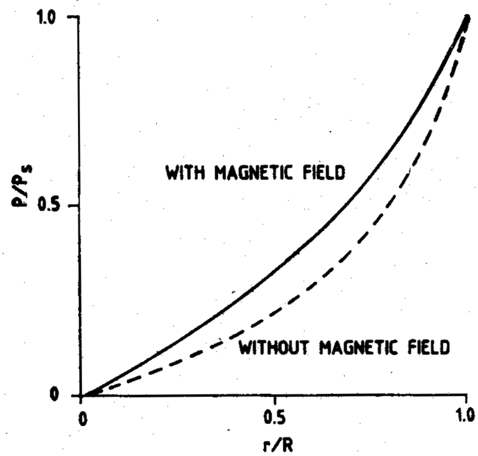


Figure 2. Adiabatic flow; density distribution.

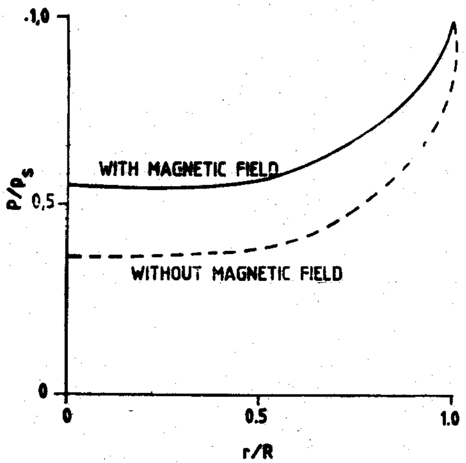


Figure 3. Adiabatic flow; Pressure distribution.

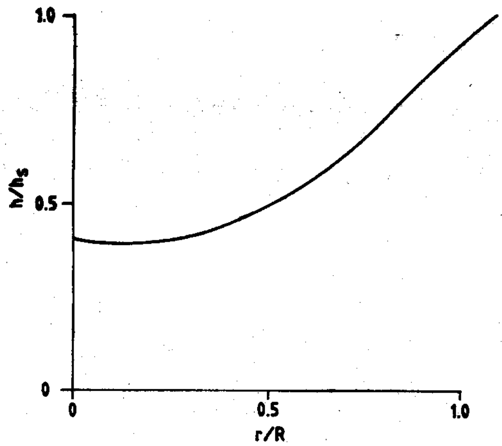


Figure 4. Adiabatic flow; magnetic field distribution.

For isothermal flows the Eqn. (7) is to be relaced by

$$\frac{\partial T}{\partial r} = 0. \tag{24}$$

The shock propagation law, that is $R \sim t^{3/5}$ is the same as in the adiabatic case. Boundary conditions at the shock front are also valid for isothermal case as given in section 2. The Eqn. (5) is evaluated approximately by replacing $\frac{\partial r}{\partial t}$, $\frac{\partial r}{\partial r_0}$ and $\frac{\partial^2 r}{\partial t^2}$ by their values at the shock front as in adiabatic case (Appendix II) which results

$$\frac{p(r_0, t)}{p_s(R)} = 1 + L(1-\xi^3) + \frac{1}{A}\left[1 - \{1 + M(1-\xi^2)\}^2\right] \tag{25}$$

where

$$L = \frac{1}{3}\left[1 + \frac{\beta^3 M_h^2}{1+\beta^2 M_h^2 (1-2\beta)}\right]^{\frac{\delta-1}{\delta}} + \frac{\beta\,[1 + 2\beta^2 M_h^2 (1-\beta)^2]}{3(1-\beta)\,[1+\beta^2 M_h^2 (1-2\beta)]} + \frac{1}{3\beta(1-\beta) M_h^2}$$

$$M = \frac{\beta}{2} + \frac{\beta}{2(1-\beta)}\left[1 + \frac{\beta^3 M_h^2}{1+\beta^2 M_h^2 (1-2\beta)}\right]^{\frac{\delta-1}{\delta}} + \frac{\beta^2\,[1+2\beta^2 M_h^2 (1-\beta)^2]}{2(1-\beta)^2\,[1+\beta^2 M_h^2 (1-2\beta)]}$$

and

$$\delta = 3/5.$$

From the Eqn. (24) and the equation of state $p = \Gamma \rho T$, where Γ is gas constant and T the temperature, we obtain

$$\frac{p}{p_s} = \frac{\rho}{\rho_s} \tag{26}$$

Evaluating the Eqn. (6), we get

$$\frac{h}{h_s} = 1 + M(1-\xi^2). \tag{27}$$

Making use of Eqn. (4), we derive a relation between reduced Eulerian λ and reduced Lagrangian ξ namely

$$\lambda^3 = 3\beta \int_0^{\xi} \frac{\xi^2\, d\xi}{1 + L(1-\xi^3) + \frac{1}{A}\left[1 - \{1 + M(1-\xi^2)\}^2\right]} \tag{28}$$

Differentiating Eqn. (28) with respect to time and making use of Eqn. (15) we obtain the expression for the velocity as

$$\frac{u}{u_s} = \frac{\beta}{(1-\beta)\lambda^2} \int_0^{\xi} \frac{[4M\xi^4\{1+M(1-\xi^2)\} - 6L\, M_h^2\, \beta^2 (1-\beta)\, \xi^5]\, d\xi}{A\,[1 + L(1-\xi^3) + \frac{1}{A}\{1 - <1 + M(1-\xi^2)>^2]^2} \tag{29}$$

Since both λ and ξ take the value 1 at the shock, the Eqn. (28) reduces to the form (22). Using the Eqns. (25) and (26), β can be found out by iterating Eqn. (22).

The expressions (25), (26), (27) and (29) form an approximate analytical solution as function of λ, which are shown in Figs. 5 to 7 by smooth curves, while the dotted curves denote the analytical solutions Ranga Rao and Ramana[6] for isothermal case.

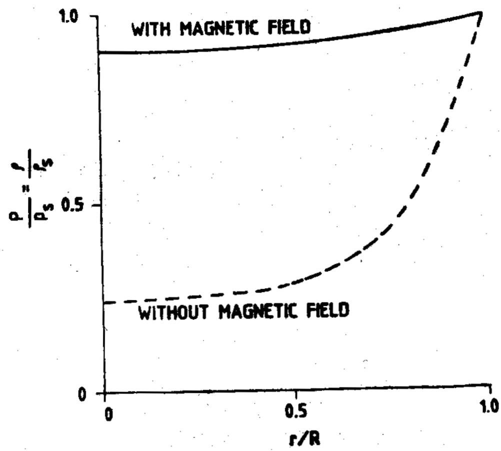
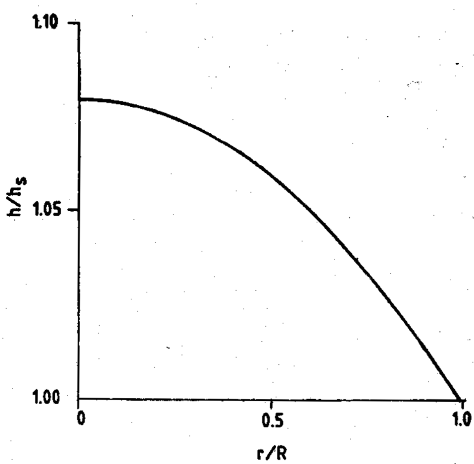


Figure 5. Isothermal flow, density and pressure distribution.



F igure 6. Isothermal flow, magnetic field distribution.

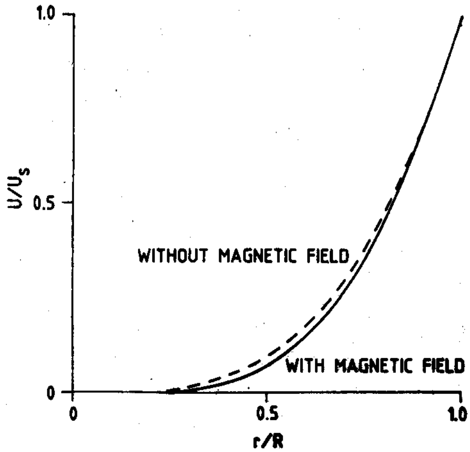


Figure 7. Isothermal flow, velocity distribution.

4. Conclusions

The presence of magnetic field increases the values of all the three quantities $\frac{u}{u_s}$, $\frac{\rho}{\rho_s}$ and $\frac{p}{p_s}$ as shown in Figs. 1-3 The magnetic field is maximum at the shock and then decreases, which becomes constant near the explosion. In isothermal case the behaviour of density and pressure are shown in Fig. 5 and they are changing abruptly with the magnetic field and show nearly constant behaviour. The magnetic field in isothermal case is increasing from the lowest i.e. 1.0 at the shock to the highest at the point of explosion. The velocity is nearly the same in the presence and absence of the magnetic field. The results obtained with the magnetic field have their significant importance in comparison to the results of the classical blast wave of Sedov[16] and Taylor[17].

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Appendix I

We first write the Taylor's expansion for $r\,(r_0, t)$ as

$$r\,(r_0, t) = R + \frac{r}{r_0}\bigg|_R (r_0 - R) + \frac{1}{2}\,\frac{\partial^2 r}{\partial r_0^2}\bigg|_R (r_0 - R)^2 + \ldots \tag{30}$$

From the continuity equation $(\rho_0\, r_0^2\, dr_0 - \rho r^2\, dr)$, we obtain

$$\frac{\partial r}{\partial r_0}\bigg|_R = \frac{\rho_0}{\rho_s} = \beta \tag{31}$$

$$\frac{1}{\rho}\,\frac{\partial \rho}{\partial r_0}\bigg|_R = \frac{2}{R}\,(1 - \beta) - \frac{1}{\beta}\,\frac{\partial^2 r}{\partial r_0^2}\bigg|_R \tag{32}$$

From the magnetic field equation $(h_0\, r_0\, dr_0 = h\, r\, dr)$, we obtain

$$\frac{1}{h}\,\frac{\partial h}{\partial r_0}\bigg|_R = (1 - \beta)\,\frac{1}{R} - \frac{1}{\beta}\,\frac{\partial^2 r}{\partial r_0^2}\bigg|_R \tag{33}$$

By differentiating Eqn. (30) with respect to time and making use of Eqn. (31), we obtain

$$\frac{\partial^2 r}{\partial t^2} = (1 - \beta)\,\ddot{R} + \frac{\partial^2 r}{\partial r_0^2}\bigg|_R \dot{R}^2 - \frac{\partial^2 r}{\partial r_0^2}\bigg|_R (r_0 - R)\,\ddot{R} \tag{34}$$

From momentum equation, we get

$$\left(\frac{\partial^2 r}{\partial t^2}\right)_s = -\left(\frac{r^2}{\rho_0\, r_0^2}\,\frac{\partial p}{\partial r_0}\right)\bigg|_R - \left(\frac{r^2}{\rho_0\, r_0^2}\, h\,\frac{\partial h}{\partial r_0}\right)\bigg|_R - \frac{h^2}{\rho r}\bigg|_R \tag{35}$$

Equating the values of $\left(\frac{\partial^2 r}{\partial t^2}\right)_s$ obtained from Eqns. (34) and (35) and also making use of Eqn. (8) and (13), we get

$$-\frac{\partial^2 r}{\partial r_0^2}\bigg|_R \dot{R}^2 = (1 - \beta)\,\ddot{R} + (1 - \beta)\,\dot{R}^2\left[\frac{1}{p}\,\frac{\partial p}{\partial r_0}\right]\bigg|_R + \left[\frac{h_0^2}{\rho_0}\left(\frac{\partial r}{\partial r_0}\right)^{-2}\frac{1}{h}\,\frac{\partial h}{\partial r_0}\right]\bigg|_R + \left[\frac{h_0^2}{\rho_0}\left(\frac{\partial r}{\partial r_0}\right)^{-1}\frac{1}{r}\right]\bigg|_R \tag{36}$$

Logarithmic differentiation of Eqn. (7) yields

$$\frac{1}{p}\frac{\partial p}{\partial r_0}\Big|_R = \left(\frac{1}{p_s}\frac{\partial p_s}{\partial r_0} + \frac{\gamma}{\rho}\frac{\partial \rho}{\partial r_0} - \frac{\gamma}{\rho_0}\frac{\partial \rho_0}{\partial r_0}\right)\Big|_R \tag{37}$$

while the shock condition Eqn. (13) gives

$$\frac{1}{p_s}\frac{\partial p_s}{\partial r_0}\Big|_R = \frac{2\ddot{R}}{\dot{R}^2}. \tag{38}$$

First from Eqns. (32), (37) and (38), calculate $\left(\frac{1}{p}\frac{\partial p}{\partial r_0}\right)\Big|_R$ and then from Eqn. (33) calculate

$$\left[\frac{h_0^2}{\rho_0}\left(\frac{\partial r}{\partial r_0}\right)^{-2}\frac{1}{h}\frac{\partial h}{\partial r_0}\right]\Big|_R \text{ and } \left[\frac{h_0^2}{\rho_0}\left(\frac{\partial r}{\partial r_0}\right)\frac{1}{r}\right]\Big|_R .$$

Now from Eqn. (34) and (36), we finally get the value of $\left(\frac{\partial^2 r}{\partial t^2}\right)_s$ as

$$\left(\frac{\partial^2 r}{\partial t^2}\right)_s = \left[1+\frac{3M_h^2\beta^3}{1+M_h^2\beta^2(\gamma-\gamma\beta-\beta)}(1-\beta)\ddot{R}\right.$$

$$+\frac{\beta}{1+M_h^2\beta^2(\gamma-\gamma\beta-\beta)}[1+2\gamma M_h^2\beta^2(1-\beta)^2\frac{\dot{R}^2}{R} \tag{39}$$

Appendix II

On exactly similar lines just as in appendix I, we can derive the value of $\left(\frac{\partial^2 r}{\partial t^2}\right)_s$ in the case of isothermal flow as

$$\left(\frac{\partial^2 r}{\partial t^2}\right)_s = \left[1+\frac{M_h^2\beta^3}{1+M_h^2\beta^2(1-2\beta)}\right](1-\beta)\ddot{R}$$

$$+\frac{\beta}{1+M_h^2\beta^2(1-2\beta)}[1+2M_h^2\beta^2(1-\beta)^2]\frac{\dot{R}^2}{R}. \tag{40}$$

The only change is that instead of differentiating Eqn. (7), we differentiate the equation $p = \Gamma\rho T$ and obtain

$$\frac{1}{p}\frac{\partial p}{\partial r_0}\Big|_R = \frac{1}{\rho}\frac{\partial \rho}{\partial r_0}\Big|_R \tag{41}$$

where the condition (24) is used.