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Magneto-Elastic Rayleigh Waves on the Surface of Orthotropic Cylinder of Varying Density

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Abstract. This paper studies magneto-elastic Rayleigh waves on the surface of orthotropic cylinder of varying density. Solving three dimensional magneto-elastic equations frequency equations for axial waves are derived.

1. Introduction

During recent years a large number of problems on interaction of elastic field with electromagnetic field have been investigated due to their extensive applications in various branches of science like Astrophysics, Geophysics, Acoustics and Plasma-Physics. While there exists a large number of papers on isotropic elastic bodies, the papers on aeolotropic elastic bodies are rare due to the various difficulties arising in the solution. Kaliski[1], Kaliski & Rogula[2], Nowacki[3], Narain & Verma[4], Narain[5,6,7], Yadava[8], and many others have discussed the propagation of magneto-elastic waves. Sequel to these, the present paper is an attempt to discuss the magneto-elastic Rayleigh waves on the surface of orthotropic cylinder of varying density. The density ρ of the material of the cylinder has been taken in the form $\rho = \rho_0 r^s$ where ρ_0 is constant and s is any integer. Such problems are very important in earth-quake researches and in the collision of elastic solids[2].

2. Fundamental Equations and Boundary Conditions

Let us consider a perfectly conducting circular cylinder, the vector of the original magnetic field being directed along the axis of the cylinder. It is assumed that the cylinder is placed in vacuum. It is also assumed that the density varies as the integral power of the radial distance. The stress strain relations for an orthotropic cylinder in cylindrical co-ordinates[9] are,

$$\sigma_{rr} = c_{11}e_{rr} + c_{12}e_{\theta\theta} + c_{13}e_{zz}$$

$$\sigma_{\theta\theta} = c_{12}e_{rr} + c_{22}e_{\theta\theta} + c_{23}e_{zz}$$

$$\sigma_{zz} = c_{13}e_{rr} + c_{23}e_{\theta\theta} + c_{33}e_{zz}$$

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$$\sigma_{rz} = c_{44} e_{rz}$$

$$\sigma_{\theta z} = c_{55} e_{\theta z}$$

$$\sigma_{r\theta} = c_{66} e_{r\theta} \tag{1}$$

where $\sigma_{rr}, \sigma_{\theta\theta},\ldots$ and $e_{rr}, e_{\theta\theta},\ldots$ etc. are components of stress and strain respectively and $c_{11}, c_{12}\ldots$ etc are elastic constants. The strain displacement relation is

$$2e_{ij} = u_{i,j} + u_{j,i} \tag{2}$$

and the stress equation of motion1 are

$$\frac{\partial}{\partial r}\sigma_{rr} + \frac{1}{r}\frac{\partial}{\partial\theta}\sigma_{r\theta} + \frac{\partial}{\partial z}\sigma_{rz} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{H^2}{4\pi}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \rho\frac{\partial^2 u}{\partial t^2} \tag{3a}$$

$$\frac{\partial}{\partial r}\sigma_{r\theta} + \frac{1}{r}\frac{\partial}{\partial\theta}\sigma_{\theta\theta} + \frac{\partial}{\partial z}\sigma_{\theta z} + \frac{2}{r}\sigma_{r\theta} + \frac{H^2}{4\pi}\cdot\frac{1}{r}\frac{\partial v}{\partial r} = \rho\frac{\partial^2 v}{\partial t^2} \tag{3b}$$

$$\frac{\partial}{\partial r}\sigma_{rz} + \frac{1}{r}\frac{\partial}{\partial\theta}\sigma_{\theta z} + \frac{\partial}{\partial z}\sigma_{zz} + \frac{1}{r}\sigma_{rz} = \rho\frac{\partial^2 w}{\partial t^2} \tag{3c}$$

The Maxwell equations governing electromagnetic field are,

$$\text{Curl } \vec{H} = 4\pi\vec{J},\ \text{Curl } \vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t},\ \text{div } \vec{B} = 0,\ \vec{B} = \mu_e\vec{H} \tag{4}$$

where the displacement current is neglected and Gaussian units have been used. We have also the Ohm's law

$$\vec{J} = \sigma\left\{\vec{E} + \frac{1}{c}\frac{\partial u}{\partial t}\times\vec{B}\right\} \tag{5}$$

In Eqns. (2), (3), (4) and (5) $\vec{H}, \vec{B}, \vec{E}, \vec{J}$ respectively stand for magnetic intensity, magnetic induction, electric intensity and current density vectors, μ_e and σ respectively, denote magnetic permeability and electric conductivity of the solid, $\vec{u}$ represents the displacement vector in the strained solid and c is the velocity of light.

The perturbation field equations are

$$\square\ \vec{E}^*, \vec{h}^* = 0 \tag{6}$$

where

$$\square = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2},\ \text{Curl } \vec{E}^* = -\frac{1}{c}\frac{\partial\vec{h}^*}{\partial t},\ \text{Curl } \vec{h}^* = \frac{1}{c}\frac{\partial\vec{E}^*}{\partial t} \tag{7}$$

Since the cylinder is a perfect conductor of electricity (i.e. $\sigma \to \infty$) the Eqn. (5) gives,

$$\vec{E} = -\frac{1}{c}\left(\frac{\partial \vec{u}}{\partial t} \times \vec{H}\right), \vec{h} = \text{Curl } (\vec{u} \times \vec{H}). \tag{8}$$

The boundary conditions on the surface $r = a$ are

$$\sigma_{ik} + T_{ik} - T^*_{ik} = 0, \vec{h} = \vec{h}^*, \vec{E} = \vec{E}^* \tag{9}$$

where σ_{ik} is mechanical and T_{ik} is Maxwellian stress-tensor.

3. Method of Solution

We assume

$$\rho = \rho_0 r^s \tag{10}$$

Where ρ_0 is constant and s is any integer. Considering the displacement components u, v, w to be function of r alone, the stress-strain relation (1) with the help of Eqn. (2) takes the form,

$$\begin{aligned}
\sigma_{rr} &= c_{11}\frac{du}{dr} + c_{12}\frac{u}{r} \\
\sigma_{\theta\theta} &= c_{12}\frac{du}{dr} + c_{22}\frac{u}{r} \\
\sigma_{zz} &= c_{13}\frac{du}{dr} + c_{23}\frac{u}{r} \\
\sigma_{rz} &= c_{44}\frac{dw}{dr} \\
\sigma_{\theta z} &= 0 \\
\sigma_{r\theta} &= c_{66}\left(\frac{dv}{ar} - \frac{v}{r}\right)
\end{aligned} \tag{11}$$

The equations in (3) as a consequence of Eqns. (10) and (11) and the substitutions $u = u_0 e^{-ipt}$, $v = v_0 e^{-ipt}$, w $= w_0 e^{-ipt}$ give three equations which with the help of the transformation

$$x = \frac{2}{s+2} r^{(s+2))/2} \tag{12}$$

take the forms

$$\frac{d^2u_0}{dx^2} + \frac{1}{x}\frac{du_0}{dx} + \left(\alpha^2 - \frac{\lambda^2}{x^2}\right)u_0 = 0 \tag{13}$$

$$\frac{d^2v_0}{dx^2} + (1 + 2\,\alpha_1)\frac{1}{x}\frac{dv_0}{dx} + \left(\beta^2 - \frac{\alpha_1^2 - \mu^2}{x^2}\right)v_0 = 0 \tag{14}$$

$$\frac{d^2w_0}{dx^2} + \frac{1}{x}\frac{dw_0}{dx} + \gamma^2 w_0 = 0 \tag{15}$$

where

$$\alpha^2 = \frac{\rho_0 p^2}{c_{11} + \frac{H^2}{4\pi}}, \quad \lambda^2 = \frac{4c_{44}}{\left(c_{11} + \frac{H^2}{4\pi}\right)(s+2)^2}$$

$$\alpha_1 = \frac{H^2}{4\pi c_{66}(s+2)}, \quad \mu^2 = \frac{H^4 - 64\pi^2 c_{66}}{16\pi^2 c_{66}^2 (s+2)^2}$$

$$\gamma^2 = \frac{\rho_0 p^2}{c_{44}}, \quad \beta^2 = \frac{\rho_0 p^2}{c_{66}} \tag{16}$$

Assuming the harmonic dependence $h_r^* = \chi_r^* e^{-ipt}$, $h_\theta^* = \chi_\theta^* e^{-ipt}$ and $h_z^* = \chi_z^* e^{-ipt}$ the Eqn. (6) with the help of Eqn. (7) gives,

$$\frac{d^2\chi_r}{dr^2} + \frac{1}{r}\frac{d\chi_r^*}{dr} + \frac{p^2}{c^2}\chi_r^* = 0 \tag{17}$$

which is also satisfied by χ_θ^* and χ_z^*

For magnetic field and Maxwellian tensors in the body, we have,

$$h_r = 0, \quad h_\theta = 0, \quad h_z = -\frac{H}{r}\frac{\partial}{\partial r}(ru)$$

$$T_{rr} = -\frac{H}{4\pi} h_z, \quad T_{r\theta} = 0, \quad T_{rz} = \frac{H}{4\pi} h_r$$

$$T_{rr}^* = -\frac{H}{4\pi} h_z^*, \; T_{r\theta}^* = 0, \; T_{rz}^* = \frac{H}{4\pi} h_r^* \tag{18}$$

The boundary conditions as a consequence of Eqn. (18) take the forms,

$$\left(c_{11} + \frac{H^2}{4\pi}\right)\frac{du}{dr} + \left(c_{12} + \frac{H^2}{4\pi}\right)\frac{u}{r} + \frac{H}{4\pi} h_z^* = 0$$

$$c_{66}\left(\frac{dv}{dr} - \frac{v}{r}\right) = 0$$

$$c_{44}\frac{dw}{dr} - \frac{H}{4\pi} h_r^* = 0$$

$$h_r = h_r^*$$

$$E_\theta = E_\theta^* \tag{19}$$

4. Axial Waves in a Finite Orthotropic Cylinder

In this case, the solutions of the Eqns. (13) to (15) must satisfy the conditions of boundedness at origin, while the solutions of the Eqn. (17) and two similar equations, at infinity. Under these conditions, we have

$$u(r) = A_1 J_\lambda \left(\frac{2\alpha}{s+2} r^{(s+2))/2} \right) e^{-ipt} \tag{20}$$

$$v(r) = 0 \tag{21}$$

$$w(r) = A_3 J_0 \left(\frac{2\gamma}{s+2} r^{(s+2))/2} \right) e^{-ipt} \tag{22}$$

and

$$h_r^*(r) = D_1 H_0^{(2)} \left(\frac{p}{c} r \right) e^{-ipt} \tag{23}$$

$$h_\theta^*(r) = 0 \tag{24}$$

$$h_z^*(r) = D_3 H_0^{(2)} \left(\frac{p}{c} r \right) e^{-ipt} \tag{25}$$

where A_1, A,, D_1, D_3 are constants, J_λ, J_0, are Bessel functions of order A and zero; and $H_0^{(2)}$ is Hankel function of order zero. and of second kind. The boundary conditions (19) with the help of Eqn. (7) and the Eqns. (20) to (25) give four equations consisting of four unknowns A_1, A,, D_1, D_3 as follows

$$A_1 \left[M_1 \left\{ \alpha a^{s/2} J_{\lambda-1} \left(\frac{2\alpha}{s+2} a^{(s+2)/2} \right) - \frac{(s+2)\lambda}{2a} J_\lambda \left(\frac{2\alpha}{s+2} a^{(s+2)/2} \right) \right\} \right.$$

$$\left. + \frac{N_1}{a} \lambda J_\lambda \left(\frac{2\alpha}{s+2} a^{(s+2)/2} \right) \right] + D_3 W_1 H_0^{(2)} \left(\frac{p}{c} a \right) = 0 \tag{26}$$

$$A, Hp\, J_\lambda \left(\frac{2\alpha}{s+2} a^{(s+2)/2} \right) + D_3 c\, H_1^{(2)} \left(\frac{p}{c} a \right) = 0 \tag{27}$$

$$A_3 a^{s/2} M_2 J_1 \left(\frac{2\gamma}{s+2} a^{(s+2)/2} \right) + W_1 D_1 H_0^{(2)} \left(\frac{p}{c} a \right) = 0 \tag{28}$$

$$A_3 J_0 \left(\frac{2\gamma}{s+2} a^{(s+2)/2} \right) - D_1 = 0 \tag{29}$$

where

$$M_1 = \left(c_{11} + \frac{H^2}{4\pi} \right),\ N_1 = \left(c_{12} + \frac{H^2}{4\pi} \right),\ W_1 = \frac{H}{4\pi},\ M_2 = c_{44}$$

Eliminating A_1, A,, D_1, D_3 from Eqns. (26) to (29), we get the frequency equation as

$$\frac{J_\lambda \left(\frac{2p}{s+2} a^{(s+2)/2} \left\{ \frac{\rho_0}{c_{11} + \frac{H^2}{4\pi}} \right\}^{\frac{1}{2}} \right)}{J_{\lambda-1} \left(\frac{2p}{s+2} a^{(s+2)/2} \left\{ \frac{\rho_0}{c_{11} + \frac{H^2}{4\pi}} \right\}^{\frac{1}{2}} \right)} =$$

$$\frac{p\, a^{(s+2)/2} \left\{ \rho_0 \left(c_{11} + \frac{H^2}{4\pi} \right) \right\}^{\frac{1}{2}}}{\left\{ c_{44} \left(c_{11} + \frac{H^2}{4\pi} \right) \right\}^{\frac{1}{2}} + \frac{H^2 ap}{4\pi c} \cdot \left(c_{12} + \frac{H^2}{4\pi} \right)} \tag{30}$$

or

$$\frac{J_1\left(\frac{2p}{s+2}\, a^{(s+2)/2}\left\{\frac{\rho_0}{c_{11}+\frac{H^2}{4\pi}}\right\}^{\frac{1}{2}}\right)}{J_0\left(\frac{2p}{s+2}\, a^{(s+2)/2}\left\{\frac{\rho_0}{c_{11}+\frac{H^2}{4\pi}}\right\}^{\frac{1}{2}}\right)} = -\frac{\frac{H}{4\pi} H_0^{(2)}\left(\frac{p}{c}\, a\right)}{\frac{2a^{s/2}}{s+2}\left(c_{11}+\frac{H^2}{4\pi}\right) p\left\{\frac{\rho_0}{c_{44}}\right\}^{\frac{1}{2}}} \tag{31}$$

We use the result

$$\lim_{x\to 0} J_n(x) \cong \frac{x^n}{2^n \pi(n)}$$

While the value of Y,(x) is always infinity at $x = 0$ and for small values of x this function is of the order of x^{-n}, if $n \neq 0$ and of the order log x if $n = 0$ (Pipes & Harvill[10]) to evaluate the frequency equation as,

$$p = \frac{2\pi c}{H^2 a}\left[\left(c_{12}+\frac{H^2}{4\pi}\right)+\left\{c_{44}\left(c_{11}+\frac{H^2}{4\pi}\right)\right\}^{\frac{1}{2}}\right] \tag{32}$$

or

$$p = \frac{Ha}{8\pi c K} \pm \frac{1}{2K}\left[\frac{H^2a^2}{16\pi^2c^2} - HK\right]^{\frac{1}{2}} \tag{33}$$

where

$$K = \frac{4\rho_0\left(c_{11}+\frac{H^2}{4\pi}\right) a^{s+1}}{c_{44}(s+2)} + \frac{Ha^2}{8\pi c^2}$$

From Eqns. (56) and (57), it is clear that the value of the frequency can be easily calculated if the values of c_{11}, c_{12}, c_{44} and H are known. Knowing the value of p, we can obtain the values of displacement components u, v, w and the strain components σ_{rr}, $\sigma_{\theta\theta}$, ...etc.

5. Axial Waves in an Iofinite Orthotropic Cylinder

In this case the solution of the Eqns. (13) to (15) must satisfy the conditions of boundedness at infinity while the solution of the Eqn. (17) and two similar equations at origin. Under these conditions, we have,

$$u(r) = B_1 Y_\lambda\left(\frac{2\alpha}{s+2}\, r^{(s+2)/2}\right) e^{-ipt} \tag{34}$$

$$v(r) = 0 \tag{35}$$

$$w(r) = B_3 Y_0 \left(\frac{2\gamma}{s+2}\; r^{(s+2)/2}\right) e^{-ipt} \tag{36}$$

and

$$h_r^*(r) = C_1 H_0^{(1)} \left(\frac{p}{c}\; r\right) e^{-ipt} \tag{37}$$

$$h:(r) = 0 \tag{38}$$

$$h,*(r) = C_3 H_0^{(1)} \left(\frac{p}{c}\; r\right) e^{-ipt} \tag{39}$$

Where B_1, B_3, C_1, C_3 are constants; Y_λ and Y_0 are Bessel functions of second kind of order λ and zero respectively; and $H_0^{(1)}$ is Hankel fun&ion of first kind of order zero. The boundary conditions given in Eqn. (19) with the help of the Eqn. (7) and the Eqns. (34) to (39) give four equations consisting of four unknowns B_1, B_3, C_1 and C_3. Elimination of these constants give the frequency equations as,

$$\frac{Y_\lambda\left(\dfrac{2p}{s+2}\; a^{(s+2)/2}\left\{\dfrac{\rho_0}{c_{11}+\dfrac{H^2}{4\pi}}\right\}^{1/2}\right)}{Y_{\lambda-1}\left(\dfrac{2p}{s+2}\; a^{(s+2)/2}\left\{\dfrac{\rho_0}{c_{11}+\dfrac{H^2}{4\pi}}\right\}^{1/2}\right)} = \frac{p a^{(s+2)/2}\left\{\rho_0\left(c_{11}+\dfrac{H^2}{4\pi}\right)\right\}^{1/2}}{\left\{c_{44}\left(c_{11}+\dfrac{H^2}{4\pi}\right)\right\}^{1/2} + \dfrac{H^2 ap}{4\pi c} - \left(c_{12}+\dfrac{H^2}{4\pi}\right)} \tag{40}$$

$$\frac{J_1\left(\dfrac{2p}{s+2}\; a^{(s+2)/2}\sqrt{\dfrac{\rho_0}{c_{44}}}\right)}{J_1\left(\dfrac{2p}{s+2}\; a^{(s+2)/2}\sqrt{\dfrac{\rho_0}{c_{44}}}\right)} = -\frac{\dfrac{H}{4\pi} H_0^{(1)}\left(\dfrac{p}{c}\, a\right)}{\left\{c_{44}\left(c_{11}+\dfrac{H^2}{4\pi}\right)\right\}^{1/2} + \dfrac{H^2 a}{4\pi c} - \left(c_{12}+\dfrac{H^2}{4\pi}\right)} \tag{41}$$

Using the results of Pipes & Harvill[10], we obtain the frequency equation as,

$$p = \frac{s+2}{2\rho_0}\; a^{(s+1)} \left[\frac{H^2}{4\pi c} \pm \left\{\frac{H^4}{16\pi^2 c^2} - \frac{8\rho_0 a^s}{s+2}\left(c_{12}+\frac{H^2}{4\pi} - \left\{c_{44}\left(c_{11}+\frac{H^2}{4\pi}\right)\right\}^{1/2}\right)\right\}^{1/2}\right] \tag{42}$$

or

$$p = i\left[\left\{\frac{H^2 a^2}{16\pi^2 c^2} + \frac{HK_1}{\pi}\right\}^{1/2} - \frac{Ha}{4\pi c}\right] \tag{43}$$

where

$$K_1 = \frac{4\rho_0\left(c_{11} + \frac{H^2}{4\pi}\right) a^{s+1}}{c_{44}(s+2)^2} - \frac{Ha^2}{4\pi c^2}$$

From Eqns. (42) and (43), we can easily obtain the value of the frequency p if c_{11}, c_{12}, c_{44} and H are known. Knowing p, we can obtain the displacement components u, v, w, and the strain components σ_{rr}, $\sigma_{\theta\theta}$, . . . etc.

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