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Time Delay Estimation Performance in a Scattering Medium

S. PRASAD

Indian Institute of Technology, New Delhi-110016.

M. S. NARAYANAN

Defence Research and Development Organization, Sena Bhavan, New Delhi-110001.

S. R. DESAI

Indian Navy.

Abstract. The effects of a scattering medium on the performance of time delay estimation are considered. The medium is assumed to exhibit angular scattering, causing angular (as well as delay) dispersion and hence loss of signal coherence across the array aperture. Both the variance (via the Cramer-Rao bound) and the bias introduced in the time-delay estimates are studied. The results have been converted to bearing and range error standard deviation and bias. It is found from these studies that there is an optimum range of values for the separation distance between the sensors in the design of an array for time delay estimation, for range and bearing measurements.

1. Introduction

The time delay estimation problem in white Gaussian noise has been extensively studied in the literature. The measurement of time delays of a signal received at several locations is particularly important for source localization[1]. A number of workers have therefore analyzed the performance of the range and bearing estimators based on the delay estimates and the geometry of the problem, in terms of variance and bias[2-4]. Some studies have also been reported on the effects of moving targets and/or platforms on the performance of time delay estimators.

The purpose of this paper is to present some results on the effects of a scattering medium on the estimation performance. For simplicity, the medium is assumed to exhibit angular cattering, scausing angular dispersion and hence loss of signal coherence across the array aperture. The performance in terms of variance is studied via the evaluation of the Cramer-Rao lower bound, which is known to be achieved by the maximum likelihood estimator presented by Carter[5]. The bias introduced by the TDE is also studied. These results have been converted to the standard deviation and bias of the resulting bearing and range estimates. Numerical results are presented for different SNR's, and sensor separation distances as a function of a 'spatial coherence loss coefficient' introduced in the sequel, for typical signal spectra and observation intervals.

It is found from these studies that there is an 'optimum' value for the separation distance in the design of an array used for time delay estimation. The optimum distance depends not only on the scattering loss coefficient, but also whether we intend to minimize the bearing error variance or the range error variance, or the corresponding biases. Fortunately, however, the performance curves are quite flat near the minima, so that a reasonable compromise is easy to obtain.

2. Model for the Scattering Medium

2.1 *Generalized Scattering Function*

In a general scattering medium, the transmission characteristics of the medium depend on space and time parameters. The five-dimensional vector may be introduced as

$$\vec{p}_i = [f_i, t_i, x_i, y_i, z_i]^T \tag{1}$$

where f_i, t_i are frequency and time parameters and (x_i, y_i, z_i) are space parameters. Thus, in the most general case, the transfer function $H(\cdot)$ depends on the vector $\vec{p}_i$.

The assumption made here is that the transfer function process is stationary in time, frequency and space. Defining the difference vector

$$\vec{p} = \vec{p}_1 - \vec{p}_2 = [\Delta f, \Delta t, \Delta x, \Delta y, \Delta z]^T \tag{2}$$

The space time correlation function of the medium can then be defined as

$$R_H(\vec{p}) = E\{H(\vec{p}_1)\, H(\vec{p}_2)\} \tag{3}$$

A dual, 'transform domain' vector $\vec{q}$ is defined as

$$\vec{q} \triangleq [\tau, \phi, u, v, w]^T \tag{4}$$

where τ = delay spread, and

ϕ = doppler spread

and where u, v and w are spreads in the angular domain and related to the direction cosines α, β, γ as shown below :

$$\left.\begin{aligned} u &= \frac{1}{\lambda}\cos\alpha \\ v &= \frac{1}{\lambda}\cos\beta \\ w &= \frac{1}{\lambda}\cos\gamma \end{aligned}\right\} \tag{5}$$

where λ is the wavelength. The generalized scattering function $L(\vec{q})$ is then obtained as the five-dimensional Fourier transform of the space time correlation function :

$$L(\vec{q}) = \int R_H(\vec{p}) \exp(j2\pi\vec{p}\cdot\vec{q})\, d\vec{p} \tag{6}$$

where $\vec{p}\cdot\vec{q}$ represents the scalar product between the vectors $\vec{p}$ and $\vec{q}$ and $d\vec{p}$ is the 5-dimensional volume element

$$d\vec{p} = [d(\Delta f), d(\Delta t), d(\Delta x), d(\Delta y), d(\Delta z)]^T \tag{7}$$

The inverse transform of $L(\vec{q})$ yields $R_H(\vec{p})$:

$$R_H(\vec{p}) = \int L(\vec{q}) \exp(-j2\pi\vec{p}\cdot\vec{q})\, d\vec{q} \tag{8}$$

The scattering function $L(\vec{q})$ describes the distribution of the signal power with respect to channel delay τ, doppler ϕ and the three angular coordinates u, v and w.

2.2 *Special Case of Angular Scattering*

Here consider a simple example of a hypothesized medium, which has only angular scattering in one direction has been considered. In other words, the space-time correlation function is assumed to depend only on the separation distance Δx, so that the vector $\vec{p}$ becomes a scalar variable Δx, and the vector $\vec{q}$ contains the scalar variable $u = \frac{1}{\lambda} \cos\alpha$. Let $\theta = 90\text{-}\alpha$, so that $u = \frac{1}{\lambda}\sin\theta$. Then

$$L(\theta) = \int R_H(\Delta x) \exp(j2\pi\Delta x \sin\theta/\lambda)\, d(\Delta x) \tag{9}$$

In order to get a clear picture of the above relation, it is instructive to obtain Eqn. (9) from elementary principles for the case of two sensors separated from each other by a distance Δx, as shown in Fig. 1.

Assume that the source is transmitting an impulse $\delta(t)$. Due to angular scattering, the two sensors receive energy from, say N, different directions, i.e. from N different

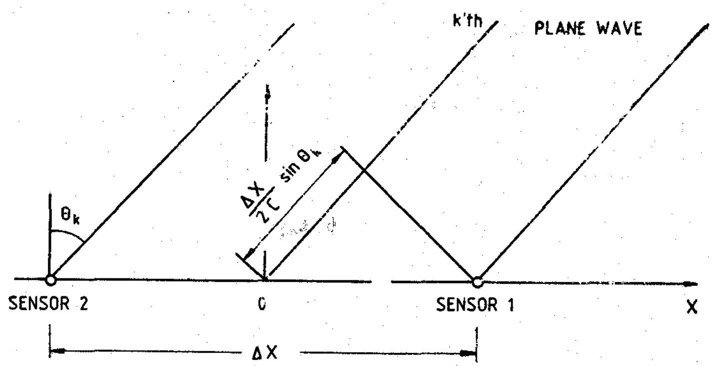

Figure 1. The array geometry for time-delay-estimation.

plane waves. The k'th plane wave arriving from direction θ_k has the associated delays $\tau_k \pm \Delta x \sin\theta_k/2c$ at the two sensors, where τ_k is the delay of the plane wave to the midpoint along the line of the two sensors. Thus the impulse response of the medium to the two sensors is given by

$$h\left((t, \pm \frac{\Delta x}{2}\right) = \sum_{k=1}^{N} a_k \delta\left(t - \tau_k + \frac{\Delta x}{2c}\sin\theta_k\right) \tag{10}$$

where a_k is the strength associated with the k'th plane wave. The corresponding transfer functions are given by

$$H\left(f, \pm \frac{\Delta x}{2}\right) = \sum_{k=1}^{N} a_k \exp\left[j2\pi f\left(-\tau_k \pm \frac{\Delta x}{2c}\sin\theta_k\right)\right] \tag{11}$$

In the limiting case, when we assume a continuum of plane waves arriving from all directions $\{-\theta_m \text{ to } \theta_m\}$ where θ_m is small, we have

$$H\left(f, \pm \frac{\Delta x}{2}\right) = \int_{-\theta_m}^{\theta_m} a(\theta) \exp\left[j2\pi f\left(-\tau(\theta) \pm \frac{\Delta x}{2c}\sin\theta\right)\right] h\theta \tag{12}$$

where $a(\theta)$ and $\tau(\theta)$ now represent the path strength and delay, respectively, associated with the plane wave arriving at an angle θ.

The space time-correlation function for this case, then becomes

$$R_H(\Delta x) = E\left[H\left(f, \frac{\Delta x}{2}\right) H^*\left(f, -\frac{\Delta x}{2}\right)\right]$$

$$= \int \int_{-\theta_m}^{\theta_m} E[a(\theta)\, a(\theta')] \exp\left(j2\pi\,[\tau(\theta') - \tau(\theta)]\right)$$

$$\exp\left[-j2\pi f\frac{\Delta x}{c}(\sin\theta' - \sin\theta)\right] d\theta'\, d\theta \tag{13}$$

Assuming, uncorrelated scattering, the equation obtained is

$$E[a(\theta)\, a(\theta') \exp\left(j2\pi f\,[\tau(\theta') - \tau(\theta)]\right) \triangleq L(\theta)\, \delta(\theta' - \theta) \tag{14}$$

which implies that it is zero when $\theta \neq \theta'$. This gives

$$R_H(\Delta x) = \int \int_{-\theta_m}^{\theta_m} L(\theta)\, \delta(\theta' - \theta) \exp\left[-\frac{j2\pi f\Delta x}{c}(\sin\theta' - \sin\theta)\right] d\theta'\, d\theta$$

$$= \int_{-\theta^m}^{\theta^m} L(\theta) \exp\left[-j2\pi f\frac{\Delta x}{c}\sin\theta\right] d\theta \tag{15}$$

which is the same as Eqn. (9), as required.

2.3 *Remarks*

(1) The one-dimensional scattering function $L(\theta)$ introduced above presents a highly simplified picture of the real medium characteristics, in that it does not reflect the loss in coherence between the signals at the two sensors (or across the array aperture, as the case may be) due to other effects of scattering, viz. spreads in the delay and doppler domains etc. However, the simplicity of the resulting model not only permits easy evaluation of its effect on the time delay estimation performance, but also makes the interpretation of the results simpler. It is, of course, of interest to generalize the results to be presented in the next section for more general scattering models.

(2) It has been shown that if the scattering surface is assumed to have a Gaussian, space-time autocorrelation function, then the resulting scattering function also has a Gaussian shape[7]. Thus, if it is assumed that

$$L(\theta) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\,\theta^2/2\sigma^2\right) \tag{16}$$

then

$$R_H(\Delta x) = \exp - \left(\frac{2\pi f}{c}\,\Delta x \sigma\right)^2 \Big/ 2 \tag{17}$$

where σ is defined to be '*spatial coherence loss coefficient*'.

It is this correlation function which is used in the next section for typical performance calculations.

3. Performance of Time Delay Estimators in Incoherent Media

3.1 *Cramer-Rao Bound for TDE*

Carter[5] has derived an expression for the variance of the time delay estimate in the neighbourhood of the true delay, for a general-cross-correlator receiver with a weighting functions $W_g(f)$ (Fig. 2). This is given by

$$\text{Var}^g\, D = \frac{\int_0^\infty |\, W_g(f) \,|^2\, G_{x_1x_2}(f)\, G_{x_2x_2}(f)[1 - C_{12}(j)]\, f^2\, df}{8\pi^2 T\, [\int_0^\infty |\, G_{x_1x_2}(f) \,|\, W_g(f) f^2\, df]^2} \tag{18}$$

where

$G_{x_1x_2}(f)$ = cross spectrum of the two signals $x_1(t)$ and $x_2(t)$ received at the two sensors (19a)

$G_{x_1x_1}(f)$, $G_{x_2x_2}(f)$ = Auto spectra of $x_1(t)$ and $x_2(t)$ respectively (19b)

$C_{12}(f)$ = Magnitude square coherency function

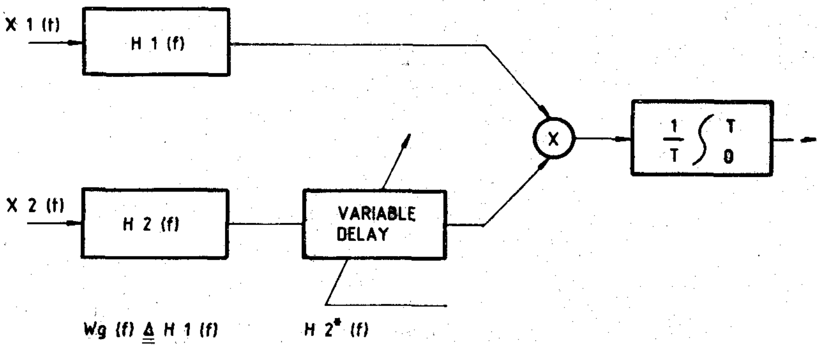

Figure 2. Generalized cross-correlator for time-delay-estimation.

$$\triangleq \left| \frac{G_{x_1 x_2}(f)}{[G_{x_1 x_1}(f)\, G_{x_2 x_2}(f)]^{1/2}} \right|^2 \tag{19c}$$

$$T = \text{observation intervel} \tag{19d}$$

In particular, it has been shown that for the maximum likelihood processor, with

$$W_{ML}(f) = \frac{C_{12}(f)}{|\, G_{x_1 x_2}(f) \,|\, [1 - C_{12}(f)]} \tag{20}$$

the variance is given (under high SNR conditions) by

$$\mathrm{Var}^{ML}\, D = \{2T \int_{-\infty}^{\infty} (2\pi f)^2\, C_{12}(f)/(1 - C_{12}(f))^2\, df\}^{-1} \tag{21}$$

which is identical with the Cramer-Rao lower bound giving the minimum obtainable variance for delay estimation. It is this bound which has been evaluated in the sequel to compare the performance in an ideal medium with that for a non-ideal or scattering medium.

It is easy to see that, for our case,

$$X_1(f) = S(f)\, H\left(f, -\frac{\Delta x}{2}\right) + N_1(f) \tag{22}$$

and

$$X_2(f) = S(f)\ \exp(-j2\pi f D)\, H\left(f, \frac{\Delta x}{2}\right) N_2(f) \tag{23}$$

where $S(f)$ is the spectrum of the desired signal, D is the delay to be estimated and $N_1(f)$ and $N_2(f)$ are the spectra of the noises in the two sensors, assumed to be statistically independent w.r.t. each other as well as with the signal of interest. It follows that

$$G_{x_1x_2}(f) = E\left\{\left[S(f)\,H\left(f, -\frac{\Delta x}{2}\right) + N_1(f)\right]\left[S(f)\right.\right.$$
$$\left.\left.\times \exp\,(j2\pi fD)\,H\left(f, \frac{\Delta x}{2}\right) + N_2(f)\right]^*\right\}$$
$$= G_{ss}(f)\exp(-j2\pi fD)\,R_H(\Delta x) \tag{24}$$

where $G_{ss}(f) \triangleq E\{\,|\,S(f)\,|\,^2\}$ is the auto spectrum of the signal.

Thus the value of the cross-spectrum $G_{x_1x_2}(f)$ and hence of the coherency function $C_{12}(f)$ would depend on the separation between the two sensors.

A close examination of Eqns. (21) and (24) clearly shows that for the scattering medium considered in the previous section, as Δx increases, $G_{x_1x_2}(f)$ decreases and the variance of the delay error increases.

3.2 Bearing and Range Error Variances

The bearing error variance is related to the time delay error variance by the relation[3]

$$\sigma_\theta = \frac{c}{\Delta x \sin\theta}\,\sigma_D \tag{25}$$

where c is the velocity of sound in the medium, σ_D is the variance of delay estimation error and θ is the true angle of the source.

For an ideal medium, therefore, the bearing error variance decreases as the separation distance between the sensors is increased, even though it is very high for small angles of the source (say $\theta \leqslant 20°$).

Since in a scattering medium, σ_D is expected to increase with separation distance due to coherence loss, it follows that there will be an optimum distance for which σ_θ is minimum. This is demonstrated in the next section, where detailed numerical results are presented for the performance.

Range can be evaluated with the help of a minimum of three sensors. Let the distance between both adjacent pairs of sensors be equal and the bearing from both the pairs be same and equal θ. Assuming that σ_D as obtained from both pairs is also same, it can be shown that[3]

$$\sigma_R^2 = \text{range error variance}$$
$$= \frac{\sqrt{2R^2c}}{(\Delta x)^2\sin^2\theta}\,\sigma_D \tag{26}$$

where R is the true range.

Once again, unlike in an ideal medium where the performance is expected to improve as the square of the separation distance, for scattering medium, a range-dependent optimum separation distance would yield the minimum value of the range error variance.

3.3 *Bias Effects*

Quazi[6] has derived expressions for the bias introduced in the bearing and range calculations from time delay measurements. The results are reproduced here for convenience :

$$\theta_B = \text{bearing bias} = -(\sigma_\theta^2 \ \cot\theta)/^2 \tag{27}$$

where θ is the true angle, and

$$R_B = \text{range bias} = \frac{\sigma_R^2}{R} \tag{28}$$

where R is the true range.

It is seen that bias in angle measurement is very high for small angles ($\theta \leqslant 20°$) and is always negative. The range bias, however, becomes significant only when the range variance is of the order of true range.

4. Typcial Performance Calculations and Numerical Results

In this section we present typical performance curves for the estimation of bearing and range from time delay measurements in scattering media. The parameters selected for study are the effects of SNR, separation distance between the sensors and the 'spatial coherence loss coefficient' introduced in section II.

The signal and noise spectra are assumed to be flat and bandlimited between 3500 Hz and 4500 Hz. It was felt that the performance of the ML estimator is mainly dependent on SNR, rather than the signal and noise spectra, hence spectrum was not taken as a parameter in these studies. The observation time is taken to be 50 sec , and has not been varied since performance is known to be directly proportional to T.

In order to provide a 'benchmark' of comparison, the performance of the maximum likelihood estimator in an ideal medium and additive Gaussian noise is first briefly summarized.

4.1 *Performance of TDE in Additive Noise in an Ideal Medium*

The important performance features of range and bearing measurements in an ideal medium, based on the formulae of the previous section, are illustrated in Figs. 3 and 4. The following observations can be made regarding the range and bearing standard deviation :

(i) The bearing standard deviation, σ_θ, varies inversely with the separation distance, even though σ_D is independent of distance in an ideal medium (Fig. 4).

(ii) The value of σ_θ also depends on the true bearing, being as low as 9.6×10^{-3} degrees for $\theta = 90°$, at an SNR of $-20\,dB$ and a separation distance of 150 meters (Fig. 3).

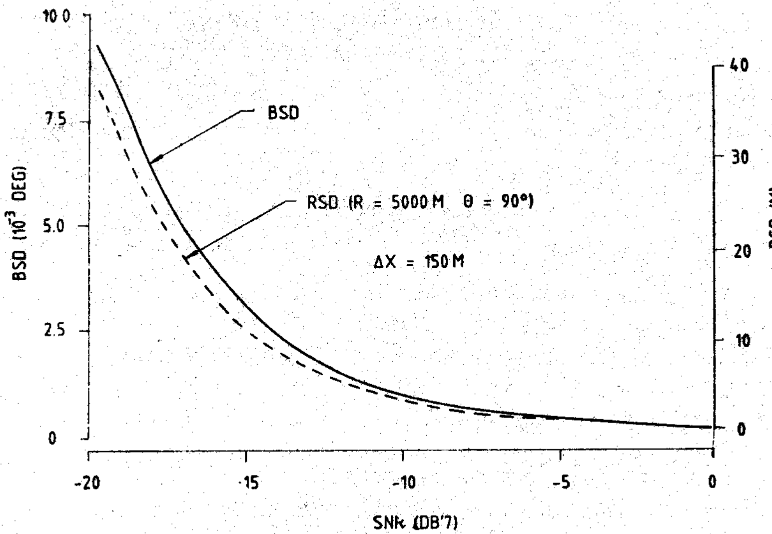

Figure 3. SNR dependence of range and bearing standard deviations : ideal medium.

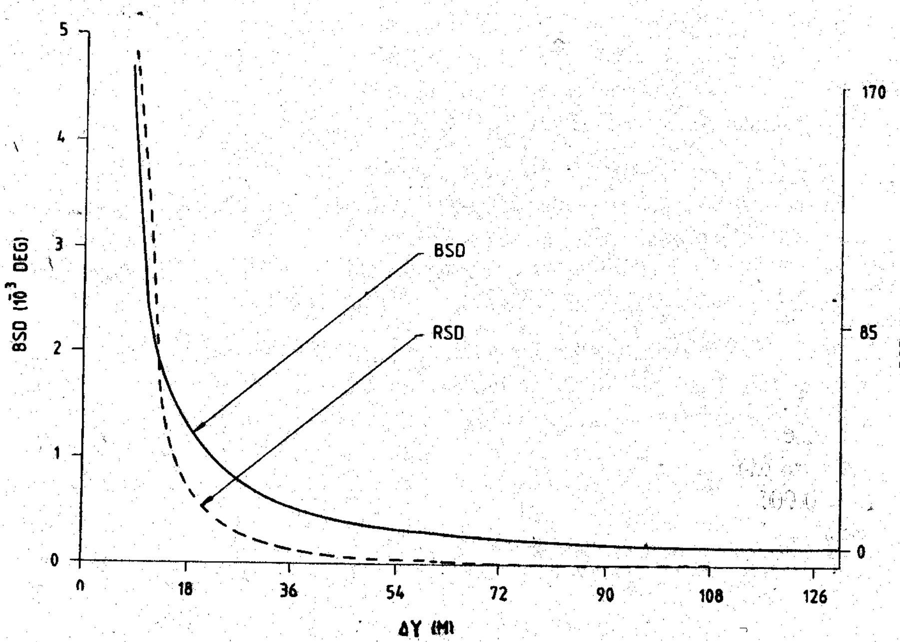

Figure 4. Dependence of range and bearing standard deviations on sensor separation distance : ideal medium.

(iii) The range standard deviation σ_R also displays a similar inverse relationship with respect to SNR and separation distance between sensors (Figs. 3 and 4 respectively).

Although no performance curves are presented here for the range and bearing bias, the following conclusions can be readily drawn from their straight forward relationships with range and bearing standard deviations :

(i) For a true bearing angle of $\theta = 90°$, the bearing bias is zero. For small angles, however, the bias is very high. The behaviour with respect to separation distance and SNR is as for the standard deviation.

(ii) The range bias is also highly dependent on the true range and bearing. The bias, like the standard deviation, is least for a true bearing of 90°.

(iii) The range bias is, once again, inversely dependent on the separation distance and SNR, being as low as 0.316 meters for a 90° bearing, —20 *dB* SNR, separation distance of 150 meters and a true range of 5000 meters.

4.2 *Performance of TDE in Scattering Media*

For the following results, the spatial correlation function of Eqn. (17), corresponding to one-dimensional angular scattering in the medium, is used for calculating the cross spectrum between the two received signals $x_1(t)$ and $x_2(t)$ via Eqn. (24). The auto spectrum is not affected because $R_H(0) = 1$. The time delay estimation error is computed via numerical evaluation of the integrals involved in Eqn. (21), which in turn is used for the computation of bearing and range error standard deviations and bias. In order to study the effect of the medium, two extrem values of σ are chosen, viz. $\sigma = 0.001$ and $\sigma = 0.01$. Figs. 5 to 8 demonstrate the important performance features of range and bearing measurement in a scattering medium. The following important observations can be made :

(i) Although the bearing error performance improves with SNR as in the ideal medium case, it is found that for equal SNR's variance in the ideal medium case is considerable less. For example, for an SNR of —20 *dB*, the bearing standard deviation is 9.6×10^{-3} degrees in an ideal medium as against 55.3×10^{-3} degrees in the scattering or incoherent medium (Fig. 5).

(ii) The bearing standard deviation (BSD) is plotted against the sensor separation distance in Fig. 6. As expected, we have an optimum separation distance for the minimum value of BSD. For the two cases shown here, i.e. $\sigma = 0.01$ and $\sigma = 0.001$, the optimum distances are 8 meters and 75 meters respectively, whereas the corresponding minimum values of BSD are 6.7×10^{-3} degrees and 0.673×10^{-3} degrees respectively. It is interesting to observe, however, that in each of these two cases, the performance remains nearly constant (equal to the optimum value) where the separation distance is varied around the optimum value.

(iii) Similar conclusions can be drawn for the range error performance, as summarized in Figs. 5 and 7. Once again, a considerable loss in SNR performance is

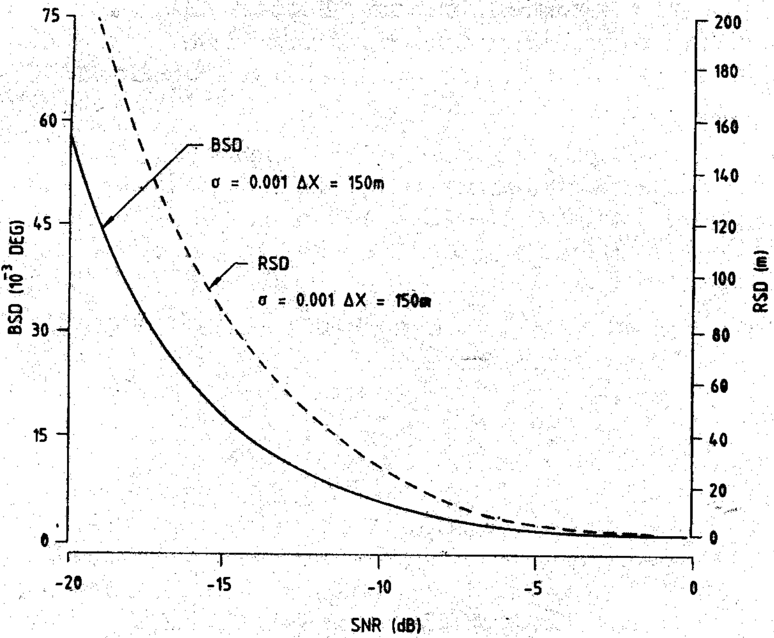

Figure 5. SNR dependence of range and bearing standard deviations : non-ideal medium.

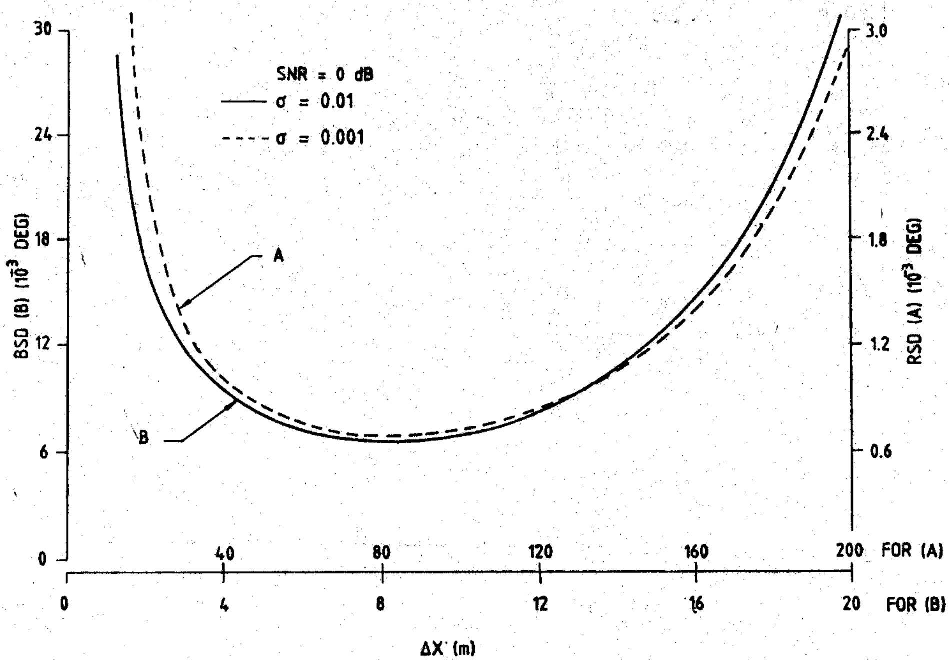

Figure 6. Dependence of bearing standard deviation on sensor separation distance : non-ideal medium.

evident for the non-ideal medium. For example with a true range $R = 5000$ meters and a separation distance of 150 meters, the range standard deviation (RSD) at $-20\ dB$ SNR is 39.7 meters in the ideal medium and 227 meters in the non-ideal medium ($\sigma = 0.001$).

Similarly the RSD curves against separation distance (Fig. 7) exhibit the existence of optimum values for minimum value. Thus the optimum separation distance is 110 meters for $\sigma = 0.001$ (with RSD $= 3.76$ meters) and 11 meters for $\sigma = 0.01$ (with RSD $= 376$ meters). Once again, however, the curves are reasonably flat around the minimum, though not as flat as for BSD.

(iv) Similar conclusions can also be seen to be applicable for bias in range measurements. The results are summarized in Fig. 8.

5. Conclusion

It can be concluded that the concept of an optimum separation distance is a very important parameter in the design of an array for range and bearing estimation based on time delay measurements. The optimum separation, however, depends not only on the scattering loss coefficient but also whether we intend to minimize the bearing error or range error standard deviation, or the corresponding biases. Hence

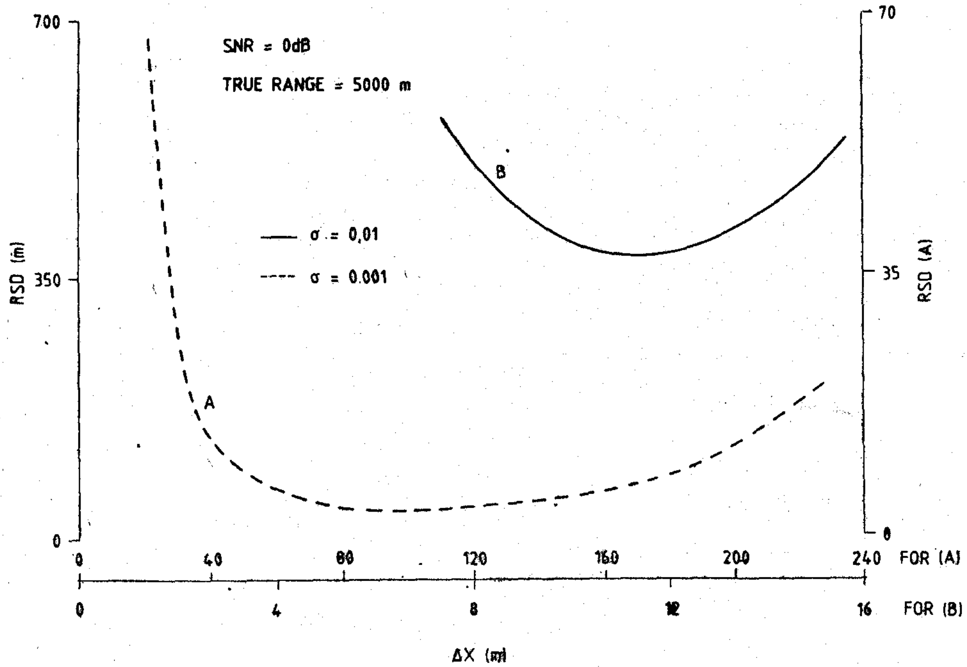

Figure 7. Dependence of range standard deviation on sensor separation distance: non-ideal medium.

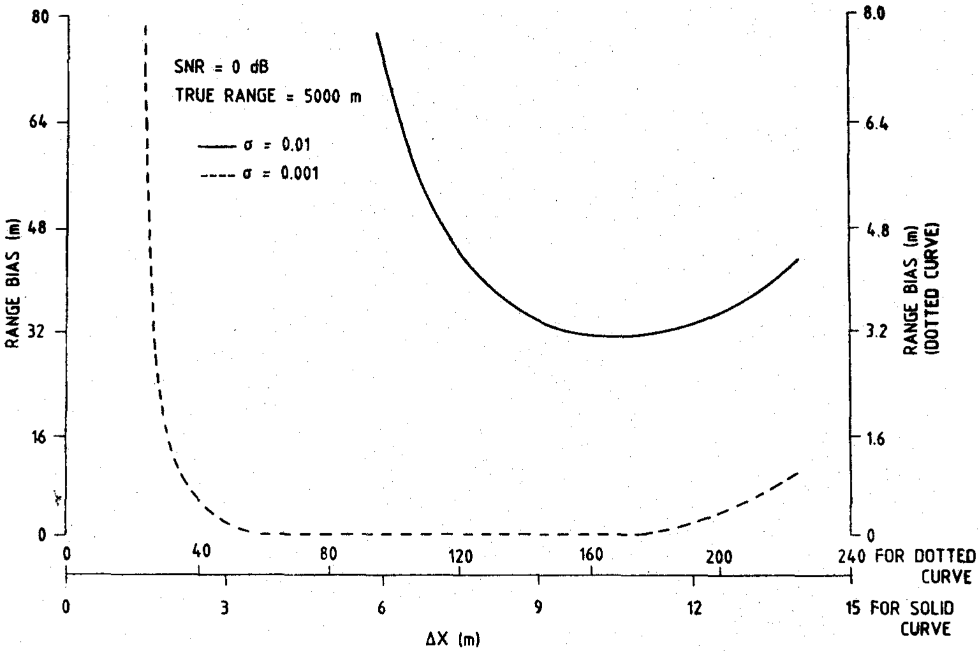

Figure 8. Dependence of range bias on separation distance.

the design of the array calls for an appropriate compromise. Fortunately, however, the performance curves are quite flat near the minima, so that a reasonable compromise is easy to obtain.

Acknowledgement

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