Def Sci J, Vol 35, No 1, January 1985, pp 65–70

A Note on the Graceful Numbering of a Class of Trees

D. R. LAKSHMI & S. VANGIPURAM
Sri Venkateswara University, Tirupati-517502.

Received 23 May 1983, revised 16 January 1984.

Abstract. An algorithm for graceful numbering of a class of trees known as T_S trees is discussed.

1. Introduction

Let T be a simple tree on n vertices $(v_1, v_2, \ldots, v_n)$ and $N_n = \{1, 2, \ldots, n\}$ be the set of n integers. T is said to be gracefully numbered if it satisfies the following three conditions.

(i) Label the vertices of T so that each vertex of T is associated with a distinct integer in N_n.

(ii) The weight of an edge of T is defined to be the absolute value of the difference between vertex numbers of its end vertices.

(iii) The weights of all edges are distinct and are in the range $[1, (n-1)]$ with the maximum value as $n-1$.

Ringel conjectured that all trees can be gracefully numbered. In this paper, we present an algorithm for gracefully numbering of a class of trees. In § 2 we deal with graceful numbering of a class of trees known as T_S trees. In § 6, graceful numbering of extended T_S trees is given. The notation and symbols followed are essentially that of Harary[1]. The problem of graceful numbering has application in communication net works. The teleprocessing tree network bears the structure of a T_S tree on n vertices, when we consider stations as vertices and links as edges of the tree. We assign a code number for each station and hence each interconnection is uniquely determined by the absolute value of the difference between the numbers assigned to its end stations. This graceful numbering helps in the conveyance of information from a station to another economically.

2. Preliminaries

We define the tree T_S as follows :

Definition The graph T_S is a tree with vertex set $v = \{v_1, v_2, \ldots, v_n\}$ with the following property :

Let $S = \{p_1, p_2, \ldots, p_m\}$ be the set of all independent paths (in the sense that they do not have a common vertex) of the tree such that these independent paths together cover all vertices in v all of which are connected by a single path p ($p \cap p_i$ has a common vertex u_i, $i = 1, 2, \ldots, m$) with vertices $\{u_1, u_2, \ldots, u_m\}$ where $u_i \in p_i$ $i = 1, 2, \ldots, m$ (called stem) satisfying the following conditions :

(i) Each path p_i has at most two vertices of degree one and all other vertices (which are not in p) are of degree two.

(ii) $2 \leqslant \deg u_i \leqslant 4$ for all $u_i \in p_i$

(Note : $\deg(u_i) = 2$, $i = 2, 3, \ldots, m - 1$, only if p_i has one vertex viz., u_i; and without loss of generality we still call p_i as a path in that case also).

(iii) The stem p divides each path p_i into at most two paths $p_i^{(1)}$, $p_i^{(2)}$ such that

$$\text{Length of } p_i^{(1)} = \text{length of } p_{i+1}^{(1)} \text{———— (a)}$$

$$i = 2, 4, 6, 8, \ldots$$

and

$$\text{length of } p_i^{(2)} = \text{length of } p_{i+1}^{(2)} \text{———— (b)}$$

$$i = 1, 3, 5, 7, \ldots$$

The structure of such a tree is illustrated in Fig.

Suppose that the stem p has m vertices $u_1, u_2, \ldots, u_m$ and the path p_i have k_i vertices so that $\sum_{i=1}^{m} k_i = n$. Denote the vertices contained in the path $p_i = p_i^{(1)} \cup p_i^{(2)}$ as

$$p_i^{(1)} = v_{i1}^{(1)}, v_{i2}^{(1)}, \ldots, v_{ik_{i1}}^{(1)}$$

and

$$p_i^{(2)} = \{ v_{i1}^{(2)}, v_{i2}^{(2)}, \ldots, v_{ik_{i2}}^{(2)} \}$$

such that $v_{ik_{i1}}^{(1)} = v_{i1}^{(2)} = u_i$ for $i = 1, 2, \ldots, m$, so that $k_{i1} + k_{i2} = k_i$.

Now we label these n vertices of the tree T_S by the elements of N_n as follows.

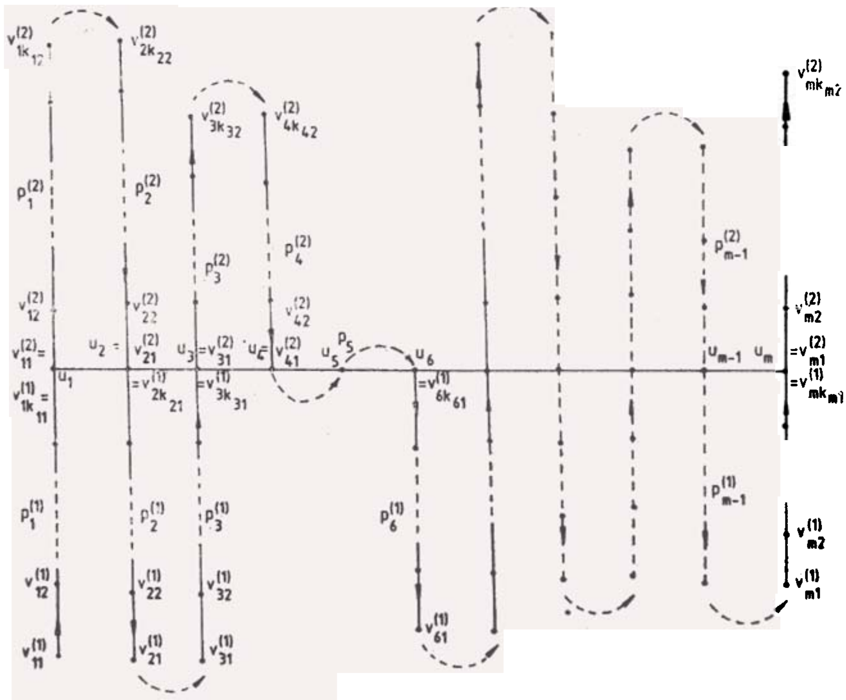

Figure 1. Structure of tree T_s.

3. Algorithm

Consider the path p_1 on k_1 vertices $v^{(1)}_{41}$, $v^{(1)}_{12}$, ... $v^{(1)}_{1k_{11}}$ $(=u_1 = v^{(2)}_{11})$, $v^{(2)}_{12}$, ...,

Label $v^{(1)}_{11}$ as n and denote this labelled vertex as $v^{(1)}_{11}(n)$. Next label $v^{(1)}_{12}$ which is adjacent to $v^{(1)}_{11}$ as 1 and denote it by $v^{(1)}_{12}(1)$, repeat the same step, successively labelling the vertices $v^{(1)}_{13}$, $v^{(1)}_{14}$, ... u_1 $(v^{(1)}_{1k_{11}} = v^{(2)}_{11})$, $v^{(2)}_{12}$, ..., $v^{(2)}_{1k_{12}}$ by the integers first by an unused maximum and then by an unused minimum of the remaining integers in N_n. When the vertices of p_1 are completely labelled, then we go from p_1 to p_2 along imaginary jumped edge according to condition (a) or (b). Continuing this procedure for the path $p_2, p_3 \dots, p_{m-1}, p_m$ either forwards or backwards according to (a) or (b) (i.e., according to the directions given on the edges of the tree in the Fig. 1). We label all the vertices with the elements of the set N_n.

4. Proof of the Algorithm

To prove the algorithm, we show that

(i) All weights of the edges are distinct, and

(ii) All labels for vertices of the tree are in the range $[1, n]$.

In the tree T_S, there are directed edges as well as some undirected edges. All directed edges constitute a directed path including some imaginary jumped-edges. This directed path spans all the vertices of T_S. There are exactly $m-1$ imaginary jumped edges and this number is the same as the number of undirected edges. It shows that the weights obtained by these undirected edges are exactly the same as the weights obtained by the imaginary jumped edges.

4. *The Weights of all Edges are Distinct*

For if possible, let the weight of an edge in p_i be the same as the weight of an edge in p_j, where $i < j$; let this edge in p_i be the join of the vertices which are labelled as a and b $(a < b)$, and let the edge in p_j with equal weight be the join of the vertices which are labelled as c and d $(c < d)$. From the algorithm, it follows that, $a<c<d<b$.

By our assumption

$$b - a = d - c$$

$$b - a + d - c = (a - c) + d < d$$

i.e., $b < d$, a contradiction.

Thus the edges in different paths are of distinct weights. In the same way, it can be easily proved that edges in the same path also have distinct weights.

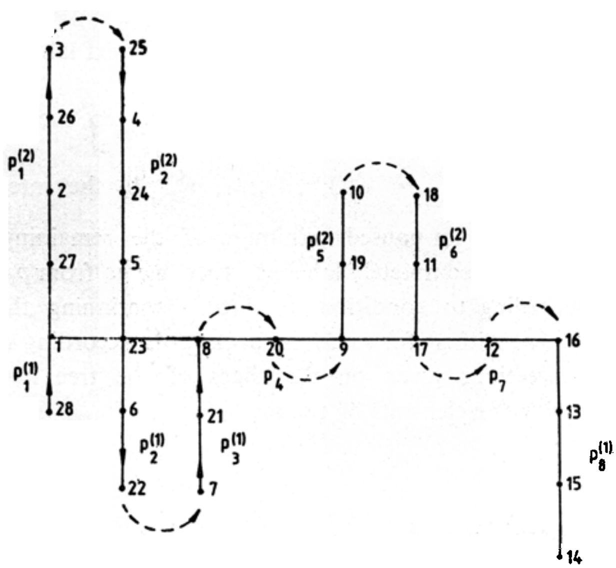

Figure 2. Structure of T_S tree with twenty eight vertices and eight independent paths.

As the number of vertices of the tree and the elements of the set N_n are equal, all labels for the vertices of T_S are in the range $[1, n]$, and no vertex remains unlabelled.

5. Conclusion

Thus all the edges in the tree T_S are having distinct weights in the range $[1, n-1]$ and the maximum weight is $n-1$. Thus T_S is gracefully numbered.

We illustrate this algorithm through an example.

Example 1 : Consider a T_S tree with twenty eight vertices and eight independent paths as shown in Fig. 2. The details of the labels given to the vertices are given in Table 1.

6. Extended T_S Tree

We define an extended T_S tree as follows :

Definition Suppose the tree T_S is extended at the vertex u_m', such that the extension at u_m has root vertex as u_m with $\deg(u_m) = l$ and the extension contains l distinct (except for the root vertex) paths of equal length.

We call this as an extended T_S tree.

Table 1.

Paths	Vertices with labels in the brackets
$p_1 = p_1^{(1)} \cup p_1^{(2)}$	$v^{(1)}_{11}(28)$, $v^{(1)}_{12}(1)$ $(= u_1(1) = v^{(2)}_{11}(1))$, $v^{(2)}_{12}(27)$, $v^{(2)}_{13}(2)$, $v^{(2)}_{14}(26)$, $v^{(2)}_{15}(3)$
$p_2 = p_2^{(1)} \cup p_2^{(2)}$	$v^{(1)}_{21}(22)$, $v^{(1)}_{22}(6)$, $v^{(1)}_{23}(23)$ $(= u_2(23) = v^{(2)}_{21}(23))$, $v^{(2)}_{22}(5)$, $v^{(2)}_{23}(24)$, $v^{(2)}_{24}(4)$, $v^{(2)}_{25}(25)$
$p_3 = p_3^{(1)}$	$v^{(1)}_{31}(7)$, $v^{(1)}_{32}(21)$, $v^{(1)}_{33}(8) = u_3(8)$
p_4	$u_4(20)$
$p_5 = p_5^{(2)}$	$u_5(9)$ $(= v^{(2)}_{51}(9))$, $v^{(2)}_{52}(19)$, $v^{(2)}_{53}(10)$
$p_6 = p_6^{(2)}$	$u_6(17)$ $(= v^{(2)}_{61}(17))$, $v^{(2)}_{62}(11)$, $v^{(2)}_{63}(18)$
p_7	
$p_8 = p_8^{(1)}$	$v^{(1)}_{81}(14)$, $v^{(1)}_{82}(15)$, $v^{(1)}_{83}(13)$, $v^{(1)}_{84}(16)$ $(= u(16))$

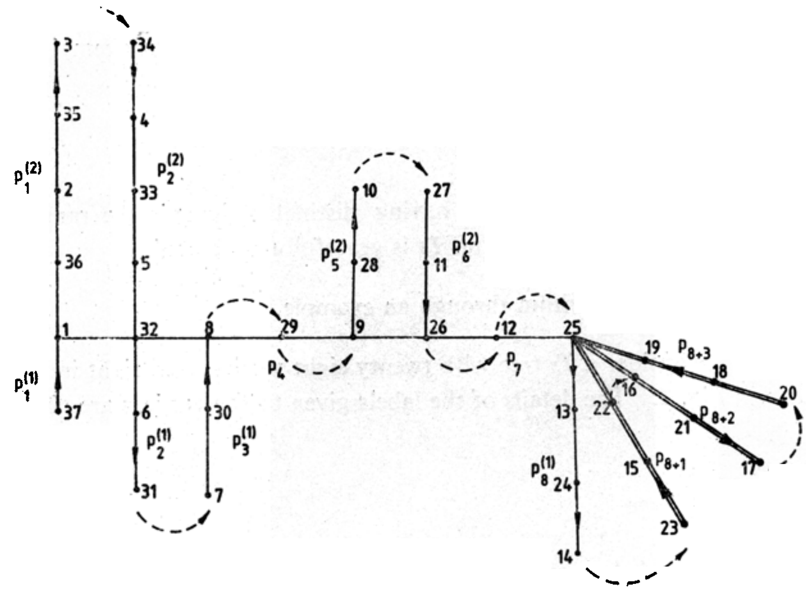

Figure 3. Structure of T_8 tree with three extended paths.

Table 2.

Extension paths	Vertices with labels in the brackets				
p_{8+1}	u_8 (25)	(= v (25)), $(8+1)_1$	v (22), $(8+1)_2$	v (15), $(8+1)_3$	v (23) $(8+1)_4$
p_{8+2}	u_8 (25)	(= v (25)), $(8+2)_1$	v (16), $(8+2)_2$	v (21), $(8+2)_3$	v (17) $(8+2)_4$
p_{8+3}	u_8 (25)	(= v (25)), $(8+3)_1$	v (19), $(8+3)_2$	v (18), $(8+3)_3$	v (20) $(8+3)_4$

This extended T_S tree also admits graceful numbering : Since the lengths of paths at u_m are the same, we follow the same procedure as given in the previous algorithm (§3) and hence it can be easily seen that this extended tree is also gracefully numbered.

This is illustrated in the following example :

Example 2 : Now we consider the extension of the T_S tree given in example (1). In this example (Fig. 3) we have three extended paths $p_{8+1}, p_{8+2}, p_{8+3}$ (= denotes the extension of T_S tree). The details of the labels given to the extended T_S tree are given in Table 2.

Reference

1. Frank Harary, 'Graph Theory', Addison-Wesley Pub. Comp., 1972.