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Spherical Location Under Restricted Distance

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Abstract . This paper deals with the problem of locating a new facility with respect to *n* given demand points on earth, with upper bounds imposed on distances between the new facility and each demand points. Distances are measured as the length of the shortest arc of great circle. The proposed algorithm makes use of a Lagrangean relaxation in which the distance constraints, which are not satisfied by the associated unconstrained solution, are incorporated in the economic function. Computational results of a limited number of test problems are presented.

1. Introduction

The interest of economists and operations researchers in the locational decision models appears to stem from the problem initiated by Alfred Weber[1] : how to locate a plant (facility) in the plane with respect to fixed markets (demand points) with the aim of minimizing the total transportation cost. The problem later studied on a net work is characterised by a solution space consisting of points on a vertex-weighted network or anywhere on the graph; the distance or time measurement is the length/time of the shortest path between nodes in the graph[2].

For optimal location on the surface of the earth where the demand points are widely separated the *planar* assumption introduces considerable errors. In a large number of logistics and location problems, such as, detection station placement, location of air/naval bases, location of emergency supply centres around the globe, and location of long range weapons systems, the distances involved occur over great expenses of the globe. Jn such situations the geodesic assumption, in which travel distance is measured as the shortest arc of a great circle, is the most appropriate approximation in modelling. Jn the past, the researchers have usually made planer assumption in dealing with large region location problems using latitude and longitude as Cartesian Coordinates[3,4]. But if demand points (destinations) are widely separated the difference between Euclidean and geodesic assumptions would be considerable, resulting in significant variations in the location of the corresponding optimal source points.

Although the interest in spherical location is relatively new, a number of literature exist. The work by Drezner and Wesolowsky[5] and Litwhiler and Aly[6] provide a conceptual discussion of, and motivation for, considering spherical location problems. Aly, Kay and Litwhiler' have discussed the theoretical results concerning the reduction of the search region and have establshed that any search for an optimal solution to minisum location problem on a sphere can be restricted to the spherically convex hull V of the set of demand points, provided V is not a great circle. However, Drezner[8] has extended this result to prove that if demand points are located on a great circle arc, so it is the optimal solution point. Katz and Cooper[9] have discussed in detail the computational aspects of the local convergence of the problem by employing a normalised gradient approach. Dhar and Rao[10,11,12] have established several properties of the problem. They have solved the single facility and the multifacility location problems involving geodesic metric and have compared these results with those of the straight line distance through the sphere.

Most of the effort in the past has been directed towards solving the unconstrained problem and how to locate the facility/facilities any where on the sphere. In doing so, one has neglected the different elements which, in the real world, effectively limit the locational possibilities of the facility/facilities. Some of these elements are : land covers only a small part of the globe, zoning regulations, international borders etc. Nothing in the literature indicates a complete work concerning this aspect of spherical looation problem. Dhar and Rao[13] have considered a spherical location problem, the solutions of which are constrained to lie within a prespecified region on the sphere. They have adopted two empirical procedures for the solution of the problem.

This paper is addressed to the problem of locating a new facility with respect to existing demand points on a sphere, with upper bounds imposed on distances between the new facility and each demand point. Distances involved are geodesic. This simple constraint on maximum distance (or time) incorporates an important notion concerning the level of utilization of a facility. One notes that attendence or utilization of a facility by users from demand sector falls with distance. Thus, the upper bounds on distances between a facility and each demand point indicates the minimal level of facility-utilization desired at each demand point. Toregas and Revelle[14] have put forward the various reasons for considering distance restrictions in the location problem in a plane. In the context of the spherical location problems, one may envision military scenarios where an air/naval base should not be too far from a number of units, in order that the base may reinforce the units in need. The method proposed for a solution is analogous to that of Schacfer and Hurter[15] for the related problem on a plane. The algorithm is based on the solution of a sequence of unconstrained problems. It uses a Lagrangean relaxation in which the distance constraints, which are not satisfied by the associated unconstrained solution, are incorporated in the economic function.

2. Problem Formulation and its Characteristics

Any point on a sphere can be defined by a two-tuple (ϕ, θ) where ϕ is the latitude $(-\pi/2 \leqslant \phi \leqslant \pi/2)$ and θ is the longitude or meridean $(-\pi < \theta \leqslant \pi)$. The shortest distance joining $r_i(\phi_i, \theta_i)$, $i = 1, 2, \ldots n$ and $r(\phi, \theta)$, via the shortest arc of the great circle is defined by $A_i(r)$ such that

$$A_i(r) = \text{Arc}\cos(\cos\phi\cos\phi_i\cos(\theta - \theta_i) + \sin\phi\sin\phi_i)$$

The single-source minisum location problem with geodesic norm on a sphere can be stated as

$$\min_{\phi,\theta} \sum_{i=1}^{n} w_i A_i(r) \tag{1}$$

where $r_i(\phi_i, \theta_i)$, $i = 1, 2, \ldots n$ are n given distinct points on the sphere with associated weights w_i and $r(\phi, \theta)$ is a point to be located on the sphere, If the new facility is restricted to be placed within specified distances of the existing demand points, then the problem may be stated mathematically as

$$\min_{\phi,\theta} \{ \sum_{i=1}^{n} w_i A_i(r) \mid A_i(r) \leqslant d_i \forall_i, i = 1, 2, \ldots n\} \tag{2}$$

where $d_i \geqslant 0$ represents the corresponding upper bounds of the distances from the possible location to destinations.

The following results, characterising the particular properties of the spherical location problem, would be useful for the subsequent development.

Definition 2. I : A set D in S_2 (S_2 being the surface of the unit sphere in Euclidean 3-space) is said to be convex if for any two points r_i and r_j in D, shortest arc distance connecting them lies entirely within the set, the convex hull of a set C in S_2 is the intersection of all convex sets containing C.

Theorem 2.1 : The geodesic distance from any given point r to any other point within a calotte (spheric disk) of radius $\pi/2$ and centre r is a convex function.

Proof: See (ref. 1 I).

Theorem 2.2 : If all the demand points in S_2 are containable within a calotte of radius $\pi/4$, and that not more than two demand points are collinear then the objective function is convex and thus possesses a unique global minimum.

Proof: See [ref. 12].

Definition 2.2 : The set $S = \{\bar{r} \mid A_i(\bar{r}) \leqslant d_i, i = 1, 2, n\}$ defines the feasible set for the constrained problem in Eqn. (2).

Definition 2.3 : The set $J = \{r_k \; A_k(r^+) > d_k, k \in i, i = 1, 2, \ldots, n\}$ defines the set of demand points whose constraints are violated by r^+ where $r+$ is the optimal solution of the associated unconstrained problem in Eqn. (1).

Proposition 2.1 : If there exists on S_2 a shortest path between any point r and r_j, $j \in i$, $i = 1, 2, \ldots, n$ that has a length not greater than d_j then the distance constraint $A_j(r) \leqslant d_j$ is redundant.

Proposition 2.2 : The constraints in Eqn (2) are consistant if

$$A_j(r_k) \leqslant d_j + d_k, 1 \leqslant j \neq k \leqslant n$$

The proof of these propositions follow from those of Schaefer and Hurter[15] who have made use of the triangular inequality property of the Euclidean metric for the related planar problem. It may be noted that the geodesic distance measurement is shown to be a metric by Blumenthal[16] and as such satisfies the triangular inequality.

The characteristics of the topology" on S_2 establish that the definition of convexity imply the connectness of any convex set. Further the important concept of 'dominance', due to Kuhn[18], holds in the case of geodesic metric[7] on S_2. In view of these as well as Theorem 2.2, it is suggested that the concept of visibility, introduced by Goldman[19,20] in convex programming[20], may be employed for characterising the solution to the constrained problem at hand.

Definition 2.4 : A point $r \in S \subset S_2$ is said to be ***visible*** from $r' \notin S$ if and only if the 'admisible' (both r' and S belong to a calotte of radius n/4) great circle arc segment $A_r(r')$ contains no point of S.

In the constrained problem in Eqn. (2), the possible locations are limited to a proper subset S of S_2. Clearly, this set is closed and compact. Hence, with the convexity property of S, following Hurter, Schaefer and Wendell[21], one may state the following result, which will be exploited in the subsequent algorithm when no solution to the associated unconstrained problem belongs to S.

Proposition 2.3 : Let r^+ be a solution to the associated unconstrained problem in Eqn. (1). Then either r^+ is a solution to the constrained problem i.e. $r^+ \in S$, or there exists a solution $\bar{r}$ to Eqn. (2) visible from r^+.

3. Solution Methodology

An approximation method due to Schaefer and Hurter[15] is adapted to solve the constrained problem at hand. The algorithm terminates when the optimal location is within $d_i + \in$ of each demand point r_i.

3.1 *Role of Lagrangean Multipliers*

The constrained problem in Eqn. (2) [can be reduced to the unconstrained problem as follows

$$\min_{\phi,\ \theta,\ \lambda_i} \left[\sum_{i=1}^{n} w_i A_i(r) + \sum_{i=1}^{n} \lambda_i (A_i(r) - d_i) \right]$$

or equivalently

$$\min_{\phi,\ \theta,\ \lambda_i} \left[\sum_{i=1}^{n} (w_i + \lambda_i) A_i(r) - \sum_{i=1}^{n} \lambda_i d_i \right] \tag{3}$$

where λ_i is the optimal Kuhn-Tucker multiplier for the ith constraint.

It is easily seen, since real argument of the Arccos in the objective function of Eqn. (2) is restricted to lie between -1 and $+1$, the function is convex in this region. Recognizing that the argument is convex and the objective function is a non negative sum of convex functions, it follows that it, too, is convex. Unfortunately though, the solution space is not convex, resulting in a non-convex programming problem[10]. Further. it is well known that the existance of a saddle point of Lagrangean function is heavily dependent upon convexity properties of underlying problem, although Arrow and Hurwicz[22] have shown that convexity assumption could be relaxed by using a modified Lagrangean approach. However, it has been noted earlier that if all demand points in the problem are located within a calotte of diameter $\pi/2$, then the objective function is unimodal within the region which assures a unique global minimum within the region. Since the generalised Slater's constrained qualification holds in all non-trivial cases, the existence of the multiplier in Eqn. (3) for this case is guaranteed by Kuhn Tucker saddle point necessary optimality theorem.

Schaefer and Hurter have interpreted these multipliers as the additional weight that would have to be added to w_i in order that the optimal solution to the constrained problem is an optimal solution to the unconstrained problem with the original weights replaced by the modified weights, ; i.e. λ_i is an additional weight to be added to the original weight so as to pull the unconstrained optimum into the feasible region S for the constrained problem. Clearly, as d_i in Eqn. (3) increases, the optimum value of the objective function for the constrained formulation decreases towards the unconstrained minimum and the artificial weights λ_i decrease. Thus the optimal value of the objective function for the constrained problem decreases strictly as the weight increases (with an exception when the facility coincide with a demand point and only that demand point's weight increases, in which case the objective function remains unchanged). The approach may be summed up in the following lemma.

Lemma : If $\boldsymbol{R}$ solves

$$\min_{\phi,\theta,\lambda_i} \sum_{i=1}^{n} (w_i + \lambda_i) A_i(r) \tag{4}$$

$$\lambda_i \geqslant 0 \ \forall\ i, \tag{5}$$

$$A_i(R) \leqslant d_i \ \forall\ i \tag{6}$$

$$A_i(R) < d_i \Rightarrow \lambda_i = 0 \tag{7}$$

then $\boldsymbol{R}$ solves also as given in Eqn. (2)

Proof: Let $\boldsymbol{r}$ be feasible for Eqn. (2). Then by Eqn. (6) and Eqn. **(7),** one has

$$\begin{aligned}
\sum_i w_i A_i(R) &= \sum_i w_i A_i(R) + \sum_i \lambda_i (A_i(R) - d_i) \\
&= \sum_i (w_i + \lambda_i) A_i(R) + \sum_i \lambda_i (-d_i) \\
&\leqslant \sum_i (w_i + \lambda_i) A_I(r) + \sum_i \lambda_i (-d_i), \text{ from Eqn. (4)} \\
&= \sum_i w_i A_i(r) + \sum_i \lambda_i (A_i(r) - d_i) \\
&\leqslant \sum_i w_i A_i(r) \quad \text{from Eqn. (5) and due to feasibility of } r
\end{aligned}$$

3.2. Search Procedure

It has been noted in Section 2 that one need only consider the demand points whose constraints are not satisfied by the optimal solution r^+ to the associated unconstrained problem. Consider the Lagrangean formulation of the constrained problem in Eqn. (2) for a demand point $r_k \in J$,

$$\min_{\phi,\theta,\lambda_k} \left[\sum_{i=1}^{n} w_i A_i(r) \ \lambda_k (A_k(r) - d_k) \right]$$

or equivalently

$$\min_{\phi,\theta,\lambda_k} \left[\sum_{\substack{i=1 \\ i\neq k}}^{n} w_i A_i(r) + (w_k + \lambda_k)(A_k(r) - d_k) \right] \tag{8}$$

An increase in λ_k is equivalent to an increase in w_k, and the objective function increases as weight $(w_k + \lambda_k)$ increases. The goal is to arrive at solution procedure based upon the successive adjustment in the λ_i's. The algorithm involves solving a sequence of unconstrained problem of Eqn. (4); i.e. for each $r_k \in J$ and finding the smallest $\lambda_k \geqslant 0$ such that the solution is feasible with respect to demand point r_k when all other weights are unchanged.

It is well known that the optimum r^+ would be located[23] at r_k, if and only if $w_k \geqslant \sum_{\substack{i=1 \\ i \neq k}}^{n} w_i$. Thus one has an upper bounds $\bar{\lambda}_k$ on λ_k for any constraining point $r_k \in J$, where $\bar{\lambda}_k = \sum_{\substack{i=1 \\ i \neq k}}^{n} w_i - w_k$. The convexity of the objective function of Eqn. (3) in interval $[0, \bar{\lambda}_k]$, enables one to apply the bisection method of Bolzano to find the optimal value of the additional weight λ_k^+ added to w_k in order to pull the unconstrained optimum into the feasible region. The details of Bolzano search given by Martos[24].

Since the objective function of Eqn (3) has the same properties as that of Eqn (2), the Lemma 3.1 indicates that a solution technique for solving unconstrained problem of Eqn. (1) might be easily extended to handle the constrained problem (Eqn. 2). The algorithm described[10] for solving the unconstrained problem in Eqn. (1) is used in conjunction with the present algorithm, with respect to each demand point $r_k \in J$, to determine whether a feasible solution to the distance constraints exists. The feasible solutions thus constructed form the feasible set S and the minimum value of the objective function in S is fixed as upper bound *UB*. The procedure is repeated once again with respect to each $r_k \in J$ and if the new solution is less than the previous *UB* it becomes the new *UB*. This procedure continues until all points in *J* have been examined and the final *UB* represents the optimum value and its corresponding solution is the optimal solution.

3.3 Algorithmic Procedure

Capitalizing on the foregoing interpretation of Lagrangian multipliers and the search procedure, the minimization algorithm is now described. The notation and terminology used is as follows :

i, j = Point indexes

k = Number of points in the set *J*. The index k essentially corresponds to only those demad points $r_k \in J$.

KOUNT = Control key used to check the feasibility criterion

p = Iteration index

ϵ = Convergence parameter ($=0.001$)

δ = Convergence parameter ($=0.0\ 1$)

n = Number of demand points

d_i = Distance constraint associated with the demand point i in radians.

w_i = Weight associated with demand point i

ϕ_i = Latitude of the demand point i in radians

θ_i = Longitude of the demad point i in radians

(.) = Value of objective function at (.)

Initial Steps

0. Input parameter (ϕ_i, θ_i) with weights w_i and distance constraints d_i, where $i = 1, 2, \ldots, n$.

1. Comupte $A_i(r_j)$, $1 \leqslant i \neq j \leqslant n$. If $A_i(r_j) > d_i + d_j$ then the constraints are inconsistant, go to 23. Otherwise, go to 2.

2. The associated unconstrained problem in Eqn. (1) can be solved by the algorithm described by Dhar and Rao[10] or by any other existing algorithm. If the terminal solution $r^+(\phi^+, \theta^+)$ does not violate the constraints then r^+ represents the optimum location and its corresponding solution is the optimal solution to problem Eqn. (2); go to 23. Otherwise, go to 3.

Main Steps

3. Input parameter ϵ and δ

4. Let $J = \{r_1, \ldots r_k, \ldots r_s\}$ be the set of demand points which violates the constraints.

5. Set KOUNT = 0

6. Set $\bar{\lambda}_k = \sum_{\substack{i=1 \\ i \neq k}}^{n} w_i - w_k, \ \forall \ r_k \in J.$

7. Set $k = 0$

8. Set $k = k + 1$

9. If both $k > |J|$ (where $\|$ denotes the cardinality of the set J) and KOUNT=0 then the feasible set S is empty, go to 23.

10. If both $k > |J|$ and KOUNT = 1 then the final UB is the optimum solution, go to 23.

11. Set $p = 1$ and $\lambda_k^{(p)} = \bar{\lambda}_k/2$

12. set $w_k = w_k + \lambda_k^{(p)}$

13. If KOUNT = 0 then go to 16. Otherwise go to 14.

14. Solve the unconstrained formulation (I), using the same algorithm as before (step 2), with respect to $r_k \in J$ and set $r(\phi, \theta) =$ the terminal solution

Compute $A_k(r)$. Set $Q(k) = A_k(r) - d_k$

15. If both $Q(k) > 0$ and $f(r) > UB$ then r_k cannot yield optimum, go to 8. Otherwise, go to 17.

16. Solve the unconstrained formulation (1), using the same algorithm as before (Step 2), with respect to $r_k \in J$ and set $r(\phi, \theta)$ = terminal solution. If r is feasible with respect to all r_i, $i = 1,\ldots,n$ then set $\bar{r} = r$ and go to 22. Otherwise, set $Q(k) = A_k(r) - d_k$ and go to 18.

17. If $|Q(k)| \leqslant \epsilon$ then set $R = r$, $UB = f(R)$ and $\lambda_k^+ = \lambda_k^{(p)}$, and go to 8, Otherwise go to 19.

18. If $|Q(k)| \leqslant \epsilon$ then set $\bar{r} = r$ and go to 22.

19. $Q(k) > \epsilon$ then perfor m Bolzano search (19 through 21) Set $\lambda_k^{(p+1)} = \lambda_k^{(p)} + \bar{\lambda}_k/2^{(p+1)}$ and go to 2 1.

20. If $Q(k) < -\epsilon$ then set $\lambda_k^{(p+1)} = \lambda_k^{(p)} - \bar{\lambda}_k/2^{(p+1)}$

21. Setp $= p + 1$;

If $|\lambda_k^{(p)} - \lambda_k^{(p-1)}| \leqslant \delta$ then go to 12. Otherwise go to 8.

22. If $k = |J|$ then set $UB = \min\limits_{\bar{r} \in S} f(\bar{r})$.

KOUNT = KOUNT + 1 and go to 7. Otherwise go to 8.

23. Stop.

4. Convergence Properties

When all the demand points lie within a calotte of radius $\pi/4$ then the objective function in Eqn. (1) is unimodal within the region'*. In this case, the algorithm has been found to converge to a solution. However, if the demand points are scattered over a larger region or, if the distance between any two demand points or between the possible location and any demand point is greater than $\pi/2$ then the algorithm will not work, because the necessity of the convexity of the objective function for the Kuhn-Tucker saddle point theorem is violated. In this case some other method must be used. Progress, in the algorithm, follows from the fact that

Table 1. Relevant data of three sample problems.

Problem		1	2	3	4	5	6	7	8	9	10
1	ϕ	38.40	34.00	41.90	52.50	59.30	39.90	55.70	34.30	23.40	39.50
	θ	-9.10	-6.50	12.50	13.40	18.90	32.80	37.70	69.10	90.20	116.20
	d	80.50	85.00	45.00	85.00	45.00	65.00	42.45	36.50	58.50	92.00
	w	0.05	0.12	0.07	0.06	0.05	0.07	0.08	0.05	0.03	0.05
2	ϕ	53.20	51.30	48.25	41.54	59.55	59.20	42.41	55.45	33.21	35.40
	θ	-6 15	-0.10	2.20	12.29	10.45	18.03	23.19	37.35	44.25	51.26
	d	45.50	39.90	25.90	22.30	25.10	8.25	20.50	22.50	50.00	52.50
	w	0.05	0.05	0.03	0.05	0.06	0.05	0.07	0.07	0.05	0.10
3	ϕ	51.50	48.90	47.40	41.90	55.70	55.70	34.40	18.90	14.60	35.60
	θ	0.40	2.30	8.50	12.50	12.60	37.70	51.40	72.80	121.00	139.70
	d	50.90	48.00	45.00	33.00	35.30	25.50	29.30	72.00	96.00	78.90
	w	0.12	0.07	0.08	0.05	0.08	0.05	0.07	0.03	0.05	0.10

ϕ, θ, *dare* in degrees.

the one dimensional Bolzano search for the optimal λ_k^+, $r_k \in J$ over the interval $[0, \bar{\lambda}_k]$ clearly converges to a unique λ_k^+, and hence converges approximately to the correct solution for each $r_k \in J$. For a further discussion on the convergence of the overall algorithm and the approach for choosing the value of the convergence parameter δ, reference can be made to the closely related procedure developed, for the planar problem, by Schaefer and Hurter[15]. The value of • can be chosen depending upon the degree of accuracy sought for the underlying problem.

5. Computational Results

The algorithm was programmed in Fortran IV and successfully tested for a limited number of sample problems. Table 1 represents the relevant data for three sample problems with ten demand points each, and the computational results are given in Table 2. Location of demand points and the corresponding values of d_i were chosen using a uniform probability distribution; the weights, w_i were generated randomly on [0, 1]. The demand points in each of the problems lie within a calotte of radius

Table 2. Computational results

Problem	Optimal (ϕ, θ) in degrees and the corresponding objective values
	$r^+ = (48.81, 19.83), f(r^+) = 14.4708$
1	$J = \{8, 9\}$
	$r = (46.53, 39.31), f(\bar{r}) = 22.9837$
	$R = (48.79, 22.50), f(R) = 15.7384$
	$\lambda_8^+ = 0.033$
	$r^+ = (49.01, 21.82), f(r^+) = 8.5996$
2	$J = \{6\}$
	$r = (57.53, 18.32), f(\bar{r}) = 9.7088$
	$R = (51.11, 20.93), f(R) = 9.0020$
	$\lambda_6^+ = 0.045$
	$r^+ = (53.11, 14.37), f(r^+) = 19.8923$
3	$J = \{7, 10)$
	$r = (43.32, 44.19), f(\bar{r}) = 26.3306$
	$R = (52.58, 17.27), f(R) = 21.5126$
	$\lambda_7^+ = 0.053$

45” that assures a global minimum. The total run time (*I*/0 included) was found to be 52 seconds on Burroughs B 6700.

6. Conclusions

It has been observed that the constrained spherical location (besides the one of restricting it to the sphere’s surface) is much more complicated then its counterpart on the plane. Due to the nature of the spherical location problem, convergence in general may be local rather than global. The Lagrangean relaxation procedure adapted has worked well for problems involving not so large regions of the globe but the procedure is bound to fail for real large region location problems due to the fact, already noted, that Kuhn-Tucker conditions for the problems are then violated. The convexity of the function defined here is arc-convexity which is different from standard convexity and, the relaxation of the convexity assumption using modified Lagrangean approach applies only to the latter one. Nothing in the literature indicates a complete work concerning this aspect of the problem and an efficient algorithm awaits a break through in nonconvex programming. Following existing methods, efforts towards developing techniques to capitalize on the structure of constrained spherical location problems could be investigated by suitably formulated search method which does not use the analytic properties of the objective functions.

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