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Exact Three Dimensional Mathematical Model for Evaluation of Antipersonnel and Antimaterial Effectiveness of Fragmenting/ Preformed Conventional Warhead

Umesh Chandra Chandola
Terminal Ballistics Research Laboratory, Chandigarh-160020.

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Abstract. The basic theory for the evalution of warhead-target interaction in three dimensional space on the concept of partitioning the target surface into sub-target surfaces having atleast two such mathematically defined boundaries which are projections of randomly fragmenting shapiro rings of the exploding shell with respect to the point of burst or preferably the centre of gravity of the shell lying on the shell-axis is presented. The equation for any given orientation of the shell-axis to the family of cones around the shell-axis has been derived for the cases: (i) when both the warhead and the target are stationary, (ii) when the warhead is moving and the target is stationary, and (iii) when both the warhead and the target are moving. Expressions for the conditional probability of kill for at least one lethal hit, and for a finite number of lethal hits, ρ, among a group of r hits $(r \geqslant \rho)$ on a tactical target by fragments/projectiles of an exploding warhead for use in the above three cases have been derived.

1. Introduction

The probability of inflicting specified damage to specified target by specified warhead of a specified missile system is called the conditional kill probability, P_k, of the warhead, provided the sub-systems of the overall missile system carrying out the function for detection, conversion, delivery and fuzing perfectly work. P_k is further a function of guidance error and fuzing error distributions, missile velocity carrying the warhead, velocity of target, angle between missile and target velocities, lethality of warhead and vulnerability of target. The condititional kill probability is expressed mathematically for known distribution of fuzing and guidance error, as

$$P_k = \sum_{i=1}^{N_k} P_i \, . \, P_{k_i} \tag{1}$$

Where P_I is the probability of burst at the i-th point near the target, it being assumed that all the points N_k are equally likely to contain a burst, and P_{k_i} is the probability of kill at the i-th point after a burst has taken place there.

Thus, in order to maximise the overall kill probability of a warhead, the ultimate aim of warhead designer is to maximise the probability of kill, P_{k_i}, by designing the warhead in such a manner that adequate number of effective fragments with optimum mass, initial velocity and angular spray are generated at burst (at the i-th point) for lethal hits on the exposed/vulnerable parts of the specified target. At present, the most widely used concept for evaluating the probability of kill, P_{k_i}, for at least one lethal hit is to calculate expected number of lethal hits from data charts of presented areas of targets for troop positions supplied by UK and emperical relationship for P_{k_i} for given mass, velocity, material of a fragment/projectile and under a military stress situation and supplied also by UK and also given in Ref. 1 (obtained from USA).

In the present paper, an exact three dimensional mathematical model is developed to work out the probability of kill, P_{k_i} for atleast one lethal hit, and for a finite number of lethal hits, ρ, among a group of r hits on a tactical target out of a large number of shooting fragments/projectiles from an exploding shell under poisson law of small probability for rare events. The formulation takes into account the parameters affecting P_{k_i}, viz the velocity and space coördinate of the centre of gravity of the warhead at burst, orientation of the axis of warhead and the solid angle subtended by the exposed surface of the target at the centre of gravity of the warhead at burst.

It may be mentioned that the amount of energy for a 'killer' or lethal hit is a function of a number of factors/parameters related to the protection level, nature and geometrical configuration of the target and that of the terrain.. Considerations of all such factors in a three dimensional mathematical model steeply increases its complexity in determining the probability of kill of a warhead, In this paper, the expression for the probability of kill for atleast one lethal hit is derived by assuming that every expected hitting fragment/projectile possesses the optimum amount of energy to inflict a 'killer' or lethal hit on the exposed/vulnerable part of howsoever small but non-zero presented area of this part of the target surface. This assumption immediately follows from the fact that the probability, $P_{k_{ij}}$ (henceforth for P_{k_i}) is a product of the probability of obtaining hit ($P_{k_{ij}}$ hit) and the probability that the hit results in a kill ($P_{k_{ij}}$ kill), the latter being unity for every hit in the treatment for atleast one lethal hit :

$$P_{k_{ij}} = P_{k_{ij}} \text{ hit } \times P_{k_{ij}} \text{ kill} \qquad (2)$$

$$P_{k_{ij}} \text{ kill} = 1 \qquad (3)$$

(The subscript j has been introduced to indicate that the notations denote quantities/ parameters pertaining to the j-th area A_j of the target surface). This explains the lethality and vulnerability criteria adopted in the formulation for atleast one lethal hit.

Now, if the nurnber of lethal hits is more than one and finite, ρ, say, among a group of r hits registered on an area of the target surface, then

$$P_{k_{ij}} \text{ kill } (P, r) < 1 \tag{4}$$

and is obtained from Bernoulli's law for r *finite,* and from poisson law for large and probability p small for a single hit to be lethal so that $r.p$ is finite. Thus, in this case Eqns. (2) and (4) hold. The necessity of the determination of probability of kill in this case is due to the fact every fragment/projectile does not possess the optimum energy require specified kill : some fragments/projectiles usually possess less and some more energy required for the specified kill.

Since the probability of kill, $P_{k_{ij}}$, is a function of expected proportion of lethal fragments, E_j, hitting an area A, of the target surface, which in turn is the product of the proportion of all the shooting fragments per unit solid angle, $f\ (\delta_{j}, _{j+1})$, in the conical zone δ_{j+1} -- δ_j between right circular cones of semi-vertical angles δ_j and δ_{j+1} (henceforth δ_j- and δ_{j+1}-cones) around the shell axis and the solid angle $\omega_{j},_{j+1}$ (say) subtended at the shell's centre of gravity (or centre of burst)by the exposed area A_j of the target surface that lies between the intersection curves formed by the intersection of δ_j and δ_{j+1} cones with the surface of the target, we proceed to derive $P_{k_{ij}}$ in the foilowing sequence :

(a) Derivation of the equation of cone of given semi-vertical angle about the axis of shell (given its direction cosines) having its vertex at the point where the direction lines of fragments' (from a Shapiro's circular ring) throw off intersect the shell axis or perferably the centre of gravity of the shell (coordinates given) with respect to a fixed coordinate system. When the coordinates of the vertex and the direction cosines of the shell axis are constants, the equation of the cone applies to cases of static detonations of shells. For a moving shell, the coordinates of the vertex and the direction cosines of the shell axis are, in general, functions of the time variable t, and the right circular cones associated with stationary shell undergo distortion. However, if the shell axis coincides with the tangent of the trajectory of the moving shell, the semi-vertical angle of the cone decreases of increases with the increase or decrease in the magnitude of the velocity of the shell maintaining the cones' right circular form around the shell axis.

(b) Transformation of the above equation of cone to moving coordinate system attached to the moving target.

By means of the cones obtained in (a) or (b), the surface of the target can be divided into a number of smaller exposed areas A_j by intersection curves formed by the intersection of successive δ_j, δ_{j+1}, δ_{j+2},. . . cones with the surface of the target. Given

the direction cosines of the normals to the smaller exposed areas, the solid angles $\omega_{j,j+1}, \omega_{j+1,j+2}, \ldots$ subtended by them at the centre of gravity or centre of burst of the shell can be calculated.

(c) Derivation of the expression for the solid angle, $\omega_{j,j+1}$, subtended at the centre of gravity of the shell by the part areas A, of the target surface covered by the conical **zones** $\delta_{j+1} - \delta_j$.

(d) Expression for the expected proportional number of lethal hits, E_j, on the area A_j of the target surface.

(e) Probability of **atleast** one lethal hit and probability of a finite number of lethal hits on the area A_j.

2. Equation to the Surface of a Cone Around a Moving Shell

On static detonation of a cylindrical shell, the fragment beams are assumed to be symmetrically distributed along sections of cones formed around the axis of the shell. The categorisation of shooting fragments is conveniently achieved if groups of fragments (emanating generally in pieces of appreciably varying masses in case of fragmenting shell) are supposed to travel over surfaces of different cones having axis of the shell as their common axis, and vertices at the centre of gravity or burst of the shell, preferably with small L/D value, becuase in accordance with Shapiro's method of obtaining static fragment pattern, fragments from different rings of a shell, in general, travel over cones having different vertices on either side of the centre of gravity depehding upon the distribution of fragment throw off angle along the shell axis. If the shell is moving, the velocity of the fragment due to static detonation gets modified in magnitude and direction. When the shell is moving with its axis tangential to its trajectory, recategorization is again possible as in the case of static detonation. In the following, we derive equation to the surface of cone when the shell is moving in space with its axis tangential to its trajectory.

Let $\bar{V}$ represent the velocity vector of the movinng shell and $\bar{U}$ the velocity vector of a fragment when the shell is statically detonated (Fig. 1). If $\bar{U}$ and $\bar{V}$ are coplaner with the shell axis and if ϕ be the angle between $\bar{U}$ and $\bar{V}$, then their magnitudes are related to the magnitude of the resultant velocity vector $\bar{V}_R$, say, of the fragment by the relation

$$V_R^2 = U^2 + V^2 + 2\,UV \cos\phi \tag{5}$$

where

$$\phi = \theta - \lambda \tag{6}$$

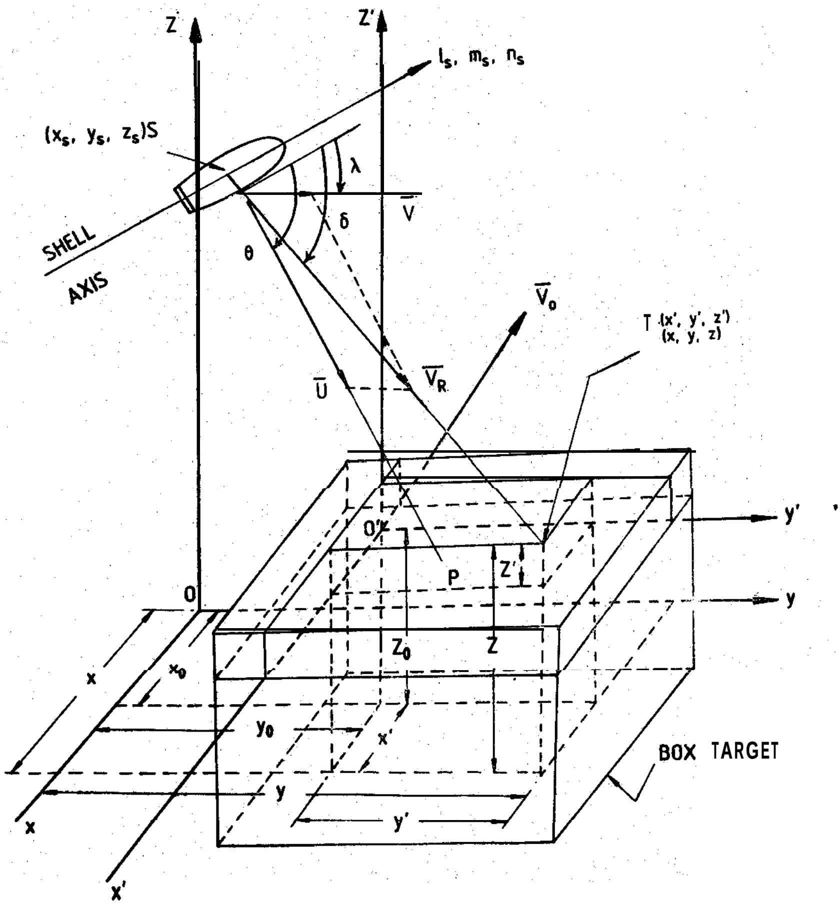

Figure 1. Oxyz is fixed coordinate system and o' x' y' z' is non-trotating moving coordinate system having velocity $\bar{V}_0$ attached to point 0 (x_0, y_0, z_0) of the moving box target. The transformation relation in Eqn. (31) between the two coordinate systems is depicted in the figure for a point *T* on the target surface. A fragment which hits point *P* of a stationary target surface on static detonation will hit point *T* of the stationary target surface when the exploding shell is in motion with velocity $\bar{V}_0$.

and θ and λ being angles which $\bar{U}$ and $\bar{V}$ respectively make with the axis of the shell. In general, when $\bar{U}, \bar{V}$ and shell axis are not coplaner, then

$$\phi = \cos^{-1} \frac{U_1 V_1 + U_2 V_2 + U_3 V_3}{UV} \tag{7}$$

where U_1, U_2, U_3 and $V_1, V_2 V_3$ are components of $\bar{U}$ and $\bar{V}$ respectively referred to a system of rectangular coordinate axes. Thus, a fragment which hits point *P* of the target surface on static detonation of the shell, will hit another point *T (x, y, z)* of the shell, will hit another point *T* (x, *y*, z) of the target surface such that *T* lies on the line of resultant velocity vector $\bar{V}_R$.

When the shell is moving with its axis tangential to its trajectory then

$$\lambda = 0, \ \phi = \theta \tag{8}$$

and, therefore,

$$V_R = \sqrt{U^2 + V^2 + 2UV \cos \theta} \tag{9}$$

and is constant for fragments of a ring of the shell. *ST* will then be a generating line of a cone around the shell axis with semi-vertical angle, δ_0, say, for then the general fragment throw off angle δ (say) becomes

$$\delta = \delta_0 \text{ when } \lambda = 0 \tag{10}$$

when *S* is the intersection point of the shell axis with the direction line *CT* of the resultant velocity ($\bar{V}_R$) of a fragment at C on the warhead casing.

As indicated above, *S* is the point at which the shell axis intersects the straight line paths of all shooting fragments from a ring of the exploding shell in accordance with Shapiro's formula' for the throw off angle, δ, of a fragment. But since the fragment spray angle is limited within a small angular zone of 20" at the centre of gravity of the shell, it is justified to take the centre of gravity of the shell as the centre of detonation for all the rings of which the full shell is supposed to be made up.

Let l_s, m_s, n_s be quantities proportional to the direction cosines of the shell axis. Then l_s, m_s, n_s are respectively proportional to V_1, V_2, V_3, the components of the velocity $\bar{V}$ of the shell in the case when the shell axis is tangential to the trajectory of the shell. If (x_s, y_s, z_s) be the coordinates of *S*, the centre of gravity of the shell, then the semi-vertical angle δ_0 of the cone over whose surface all the fragments of a ring would move is given by

$$\cos \delta_0 = \frac{l_s (x - x_s) + m_s (y - y_s) + n_s (z - z_s)}{\sqrt{l_s^2 + m_s^2 + n_s^2}\,\sqrt{(x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2}} \tag{11}$$

$$\text{or} = \frac{V_1 (x - x_s) + V_2 (y - y_s) + V_3 (z - z_s)}{V \sqrt{(x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2}} \tag{12}$$

which, on rationalisation and rearrangement, gives

$$a (x - x_s)^2 + b (y - y_s)^2 + c (z - z_s)^2 + 2f (y - y_s)(z - z_s) + 2g (z - z_s)(x - x_s) + 2h (x - x_s)(y - y_s) = 0 \tag{13}$$

where

$$\left.\begin{aligned} a &= l_s^2 - \left(l_s^2 + m_s^2 + n_s^2\right)\cos^2\delta_0,\ b = m_s^2 - \left(l_s^2 + m_s^2 + n_s^2\right)\cos^2\delta_0 \\ c &= n_s^2 - \left(l_s^2 + m_s^2 + n_s^2\right)\cos^2\delta_0, \\ f &= m_s n_s,\ g = n_s l_s,\ h = l_s m_s \end{aligned}\right\} \tag{14}$$

Putting

$$X = x - x_s,\ Y = y - y_s,\ Z = z - z_s \tag{15}$$

in Eqn. (13), that is, shifting the origin to S (x_s, y_s, z_s), it becomes

$$aX^2 + bY^2 + cZ^2 + 2fYZ + 2gZX + 2hXY = 0 \tag{16}$$

For a constant value of δ_0, i.e. for all those fragments whose throw off angle is same, Eqn. (16) represents a right circular cone with vertex at the origin provided

$$abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0 \tag{17}$$

Thus, Eqn. (16) and, therefor, Eqn (13) represents a cone on whose surface only those fragments travel for whom the angle of throw δ_0 is the same.

Now, for finding the equation of the cone, we must know the value of the throw off angle δ for given values of U, V, V_R, θ and λ concerning the fragments and the shell. The general throw off angle δ is thus a function of U, V, V_R, θ and λ. A relation between them can be found in the following way.

Let $\hat{i}, \hat{j}, \hat{k}$ be unit vector parallel to the axes x, y, z respectively, and let the components of $\bar{V}_R$, $\bar{U}$, $\bar{V}$ be given by the following :

$$\left.\begin{aligned} \bar{V}_R &= V_{1R}\,\hat{i} + V_{2R}\,\hat{j} + V_{3R}\,\hat{k} \\ \bar{U} &= U_1\,\hat{i} + U_2\,\hat{j} + U_3\,\hat{k} \\ \bar{V} &= V_1\,\hat{i} + V_2\,\hat{j} + V_3\,\hat{k} \end{aligned}\right\} \tag{18}$$

$$V_{1R} = U_1 + V_1,\ V_{2R} = U_2 + V_2,\ V_{3R} = U_3 + V_3 \tag{19}$$

Let the direction cosines of V_R be L, M, N, of U be L_u, M_u, N_u and of V be L_v, M_v, N_v.

Thus

$$V_{1R} = V_R L,\ \text{etc.},\ U_1 = UL_u,\ \text{etc.},\ V_1 = VL_v,\ \text{etc.} \tag{20}$$

On using these in Eqn. (19), we get

$$L = \frac{UL_u + VL_v}{V_R},\ M = \frac{UM_u + VM_v}{V_R},\ N = \frac{UN_u + VN_v}{V_R} \tag{21}$$

Now, from Fig. 2,

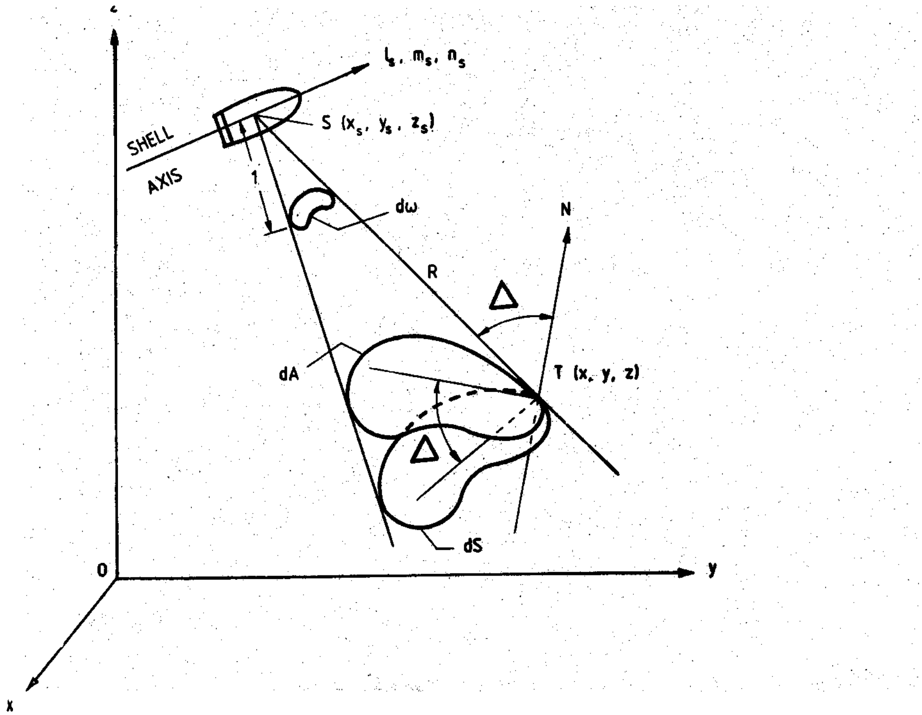

Figure 2. Solid angle $d\omega$ subtended by elementary surface dA of a target surface at the centre of gravity S of the shell.

$$\cos \delta = (l_s L + m_s M + n_s N) \Big/ \sqrt{l_s^2 + m_s^2 + n_s^2} \tag{22}$$

Eliminating $L, M,$ N between Eqns. (21) and (22), we get

$$\cos \delta = \frac{U[l_s L_u + m_s M_u + n_s N_u] + V[l_s L_v + m_s M_v + n_s N_v]}{V_R \sqrt{l_s^2 + m_s^2 + n_s^2}} \tag{23}$$

$$= \frac{U \cos \theta + V \cos \lambda}{V_R} \tag{24}$$

which states that the component of fragment's resultant velocity along the shell axis is the sum of the components of fragment's and shell velocities along the shell axis although the shell axis does not necessarily lie in the plane of the velocity vectors. For the case in which the shell axis coincides with the tangent to the trajectory of the moving shell, Eqn. (24) gives

$$\cos \delta_0 \; \frac{U \cos \theta + V}{V_R} \tag{25}$$

where $\bar{V}_R$ is now given by Eqn. (9). In Eqn. (24), $\bar{V}_R$ is given by Eqns (5) and (7).

Now, since the shell is moving with the velocity $\bar{V}$ in space, the coordinates (x_s, y_s, z_s) of its centre of gravity, S, are made up of two parts : (i) initial coordinates of S; and (ii) the increments in the initial coordnates due to the velocity of the shell. Thus, we can write

$$x_s = x_{os} + x_1 (t), \; y_s = y_{os} + y_1 (t), \; z_s = z_{os} + z_1 (t) \tag{26}$$

where (x_{os}, y_{os}, z_{os}) is the initial position of S, and

$$\left.\begin{aligned} x_1 (t) &= \int_0^t V_1 (t)\, dt, \; y_1 (t) = \int_0^t V_2 (t)\, dt, \; z_1 (t) = \int_0^t V_3 (t)\, dt \\ x_1 (0) &= y_1 (0) = z, (0) = 0 \end{aligned}\right\} \tag{27}$$

since the components V_1, V_2, V_3 vary with time t as the centre of gravity, S, of the shell moves along the trajectory defined by the parameteric equations (27).

Thus, from Eqns. (13) and (27) at any instant of time t, the equation of the moving right circular cone of semi-vertical angle given by Eqn. (25) around the shell axis with direction cosines l_s, m_s, n_s or $(V_1, V_2, V_3)/V$ and having the point $(x_{os} + x_1, y_{os} + y_1, z_{os} + z_1)$ as its vertex is given by

$$\begin{aligned} & a (x - x_{os} - x_1) + b (y - y_{os} - y_1) + c (z - z_{os} - z_1) \\ & \quad + 2f (y - y_{os} - y_1)(z - z_{os} - z_1) + 2g (z - z_{os} - z_1)(x - x_{os} - x_1) \\ & \quad + 2h (x - x_{os} - x_1)(y - y_{os} - y_1) = 0 \end{aligned} \tag{28}$$

This equation represents a cone with axis moving velocity $\bar{V}$ along the direction of the tangent to its trajectory. Any numerical value of t when put in the expressions for x_1 (t), y_1 (t), z_1 (t) occurring in Eqn. (28) uniquely defines the position of the cone in space at that instant of time from its initial position when the coordinates of its vertex were (x_{os}, y_{os}, z_{os}).

3. Case of Static Detonation of Shell

In this case $V = 0$, and, therefore, from Eqn. (9), $V_R = U$. Hence from Eqn. (25), we get

$$\delta_0 = \theta \tag{29}$$

which when substituted in Eqn. (1 3) gives the equation to the cone for fragments traversing over the surface of the cone at the same throw off angle θ given by Shapiro's formula'

$$\tan(\phi_1 - \theta) = \frac{U}{2V_D} \cos\left(\frac{\pi}{2} \phi_2 - \phi_1\right) \tag{30}$$

where ϕ_1 is the angle which the normal at a point to the inner surface of the shell makes with the axis of the shell, and ϕ_2 is the angle which the normal to the detonation wave front makes with the axis at this point of warhead casing; V_D is the velocity of detonation.

4. Case when the Target is also Moving

In the case of a moving target, the equation to a cone can be derived in a more useful and simpler form by fixing the coordinate system to a point of the moving target. The new coordinate system thus acquires the speed and direction of the moving target with respect to which the coordinates of the centre of gravity, S and the direction cosines (or quantities proportional to them) of the axis of the shell are required to be known or defined. The formulation becomes more simpler if the orientation of the axis remains unaltered with the rotation of the moving target, for example, when negotiating bends or skipping direct projectile hit, although the origin remains fixed to the same point of the target.

Let the velocity vector $\bar{V}_0$ having components V_{01}, V_{02}, V_{03} denote the velocity of the moving target with respect to the fixed coordinate system used in the derivation of equation (28) of the moving cone. Let x', y', z' be current coordinate with respect to a non-rotating moving coordinate system attached to a point of the target. The following transformation relations hold for coordinates of any point *Tin* the fixed and non-rotating moving coordinate system as shown in Fig. 1 :

$$x = x' + x_0(t),\ y = y' + y_0(t),\ z = z' + z_0(t) \tag{31}$$

where

$$\left.\begin{aligned} x_0(t) = \int_0^t V_{01}(t)\,dt,\ y_0(t) = \int_0^t V_{02}(t)\,dt,\ Z_0(t) = \int_0^t V_{03}(t)\,dt \\ x_0(0) = y_0(0) = z_0(0) = 0 \end{aligned}\right\} \tag{32}$$

The second relations in Eqn. (32) when used in Eqn (31) show that the two coordinate systems are initially coincident. Obviously the target moves along the trajectory given by the parametric equation (32).

Transforming Eqn. (28) to non-rotating moving coordinate system x', y', z' by means of the relations (3 1), we get

$$a\,[x' - \{x_{0s} + (x_1 - x_0)\}]^2 + b\,[y' - \{y_{0s} + (y_1 - y_0)\}]^2 + c\,[z' - \{z_{0s} + (z_1 - z_0)\}]^2$$
$$+ 2f\,[y' - \{y_{0s} + (y_1 - y_0)\}]\,[z' - \{z_{0s} + (z_1 - z_0)\}]$$
$$+ 2g\,[z' - \{z_{0s} + (z_1 - z_0)\}]\,[x' - \{x_{0s} + (x_1 - x_0)\}]$$
$$+ 2h\,[x' - \{x_{0s} + (x_1 - x_0)\}]\,[y' - \{y_{0s} + (y_1 - y_0)\}] = 0 \tag{33}$$

as the equation to the moving cone with respect to the moving target. Equation (33) clearly brings out the effect of the relative velocity $\bar{V} - \bar{V}_0$ of the shell and therefore of the cone with respect to the moving frame of reference on the initial position (x_{0s}, y_{0s}, z_{0s}) of the centre of gravity of the shell by increamenting it by $x_1 - x_0$, $y_1 - y_0$, $z_1 - z_0$ in time t corresponding to relative velocity components, $V_1 - V_{01}$, $V_2 - V_{02}$, $V_3 - V_{03}$ at time t. Thus, the coordinates of the centre of gravity of the shell (i.e. of the vector of the cone) at time t will be $(x_{0s} + x_1 - x_0, y_{0s} + y_1 - y_0, z_{0s} + z_1 - z_0)$ referred to coordinate system fixed to the moving **target.** It may be emphasised that a, b, c, f, g, h are values at time t of burst when calculating the probability of kill of the warhead.

5. Solid Angle Subtended by Target Surface at the Ceutre of Gravity of the Shell

Let $Oxyz$ be a fixed system of rectangular coordinates with respect to which the coordinates of the centre of gravity of the shell are (x_s, y_s, z_s), (Fig. 2). Let $T\,(x,y,z)$ be any point on the periphery of an elementary area dA of the surface of the target. If $d\omega$ be the solid angle subtended by the area dA at S, and if dS is the projection of dA on a sphere with centre at S and radius $ST = R$ (say), then

$$d\omega = \frac{dS}{R^2} = \frac{\cos\Delta\,.\,dA}{R^2} \tag{34}$$

where Δ is the angle between the dA and dS.

Lt l, m, n be quantities proportional to the direction cosines of the normal TN to the surface dA. Since the direction cosines of ST are proportional to $x - x_s$, $y - y_s$, $z - z_s$, we have

$$\cos\Delta = \frac{l\,(x - x_s) + m\,(y - y_s) + n\,(z - z_s)}{\sqrt{l^2 + m^2 + n^2}\,\sqrt{(x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2}} \tag{35}$$

Since

$$R = ST = \sqrt{(x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2} \tag{36}$$

by Eqns. (34) (35) and (36), we get

$$d\omega = \frac{[l\,(x - x_s) + m\,(y - y_s) + n\,(z - z_s)]\,dA}{\sqrt{l^2 + m^2 + n^2}\,[(x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2]^{3/2}} \tag{37}$$

for the case when both the detonating shell and the target are stationary. In Eqn. (37), l, m, n are known quantities if the equation to the target surface is given :

$$l = \frac{\partial F}{\partial x}, m = \frac{\partial F}{\partial y}, n = \frac{\partial E}{\partial z} \tag{38}$$

where

$$f(x, y, z) = 0 \tag{39}$$

is the equation of the surface of the target.

In the case of stationary target and moving shell, the solid angle subtended by the elementary area dA of the target surface at the instant of burst (or at any time) at the moving centre of burst, S, is given by

$$d\omega = \frac{[l(x - x_{0s} - x_1) + m(y - y_{0s} - y_1)\, n(z - z_{0s} - z_1)]\, dA}{\sqrt{l^2+m^2+n^2}\, [(x-x_{0s}-x_1)^2 + (y-y_{0s}-y_1)^2 + (z-z_{0s}-z_1)^2]^{3/2}} \tag{40}$$

since in this case

$$ST = R = [(x - x_{0s} - x_1)^2 + (y - y_{0s} - y_1)^2 + (z - z_{0s} - z_1)^2]^{1/2} \tag{41}$$

which is the distance between the points $S\,(x_{0s} + x_1, y_{0s} + y_1, z_{0s} + z_1)$ and $T\,(x,y,z)$.

In the case when the target is also moving, the quantities l, m, n vary, in general, with time t. If l, m, n are values at time t or at the instant of burst, then for a moving target and moving shell, the solid angle subtended by dA at S at time t is given by

$$d\omega = \frac{\Big[l\,[x' - \{x_{0s} + (x_1 - x_0)\}] + m\,[y' - \{y_{0s} + (y_1 - y_0)\}] + n\,[z' - \{z_{0s} + (z_1 - z_0)\}]\Big]\, dA}{\sqrt{l^2+m^2+n^2}\Big[[x' - \{x_{0s} + (x_1 - x_0)\}]^2 + [y' - \{y_{0s} + (y_1-y_0)\}]^2 + [z' - \{z_{0s} + (z_1 - z_0)\}]^2\Big]^{3/2}} \tag{42}$$

since here

$$ST = R = \Big[\Big[[x' - \{x_{0s} + (x_1 - x_0)\}]^2 + [y' - \{y_{0s} + (y_1 - y_0)\}]^2 + [z' - \{z_{0s} + (z_1 - z_0)\}]^2\Big]\Big]^{1/2} \tag{43}$$

which is the distance between the points $S\,(x_{0s} + x_1 - x_0, y_{0s} + y_1 - y_0, z_{0s} + z_1 - z_0)$ and $(T\; x', y', z')$.

6. Expected Number of Lethal Hits on the Exposed Area of Target Surface

The expected proportion of lethal fragments hitting the elementary area dA can be calculated after determining the value of the solid angle from eqns (37), (40) (42) in accordance with the tactical situation under consideration, provided the proportion of all the bomb's fragments per unit solid angle in the conical zone covering the area dA is known.

If $f(\delta_{j,j+1})$ be the proportion of all bomb's fragments per unit solid angle in the conical zone $\delta_{j+1} - \delta_j$, then the exposed proportion of lethal fragments per unit solid angle, $E_{j,j+1}$, hitting the expected area A_j of the target surface that lies between the intersection curves formed by the δ_j and δ_{j+1} cones with the target surface, is given by

$$E_{j,j+1} = f(\delta_{j,j+1}) \int_{A_j} d\omega_{j,j+1} \tag{44}$$

where

$$f(\delta_{j,j+1}) = \frac{\sigma_j}{N\, 2\pi (\cos \delta_j - \cos \delta_{j+1})} \tag{45}$$

and, from Eqn. (42), for the general case (iii) of target and missile warhead both moving, the later moving tangentially to its trajectory,

$$d\omega_{j,j+1} = \frac{\Big[l_j [x' - \{x_{0s} + (x_1 - x_0)\}] + m_j [y' - \{y_{0s} + (y_1 - y_0)\}] + n_j [z' - \{z_{0s} + (z_1 - z_0)\}] \Big] dA_j}{\sqrt{l_j^2 + m_j^2 + n_j^2} \Big[[x' - \{x_{0s} + (x_1 - x_0)\}]^2 + [y' - \{y_{0s} + (y_1 - y_0)\}]^2 + [z' - \{z_{0s} + (z_1 - z_0)\}]^2 \Big]^{3/2}} \tag{46}$$

in which l_j, m_j, n_j are quantities proportional to the direction cosines of the normal to the elementary area dA_j and determined by Eqn. (38), σ_j is the number of fragments for all weight groups in the conical zone $\delta_{j+1} - \delta_j$ and N is the total number of fragments of the shell. Clearly, for the tactical situation (i), $d\omega_{j,j+1}$ can be obtained from Eqn. (46) by putting $x_0 = y_0 = z_0 = 0$ and also $x_1 = y_1 = z_1 = 0$. For the tactical situation (ii), $d\omega_{j,j+1}$ is obtained by putting $x_0 = y_0 = z_0 = 0$ in Eqn. (46)

7. Probability of Atleast One Lethal Hit and Probability of Finite Number of Lethal Hits

Since E_j (henceforth written for $E_{j,j+1}$) is the mean expected number of hits on the exposed area A_j out of a large number of shooting fragments in the conical zone $\delta_{j+1} - \delta_j$, the probability of never hitting the area A_j, by Poisson low of small probabilities, is exp $(-E_j)$. The probability that atleast one lethal hit will be registered on the exposed area A_j, is, therefore,

$$P_{k_i j}\ (r \geqslant 1) = 1 - \exp\ (-E_j) \tag{47}$$

where E_j is known from Eqn. (44)

The probability of not more than $(r - 1)$ hits is

$$P_{k_i j}\ (\leqslant r - 1) = \sum_{r=0}^{r-1} \frac{(E_j)^r}{r!} \exp\ (-E_j) \tag{48}$$

Therefore probability of atleast r hits is

$$P_{k_i j}\ (\geqslant r) = 1 - \sum_{r=0}^{r-1} \frac{(E_j)^r}{r!} \exp\ (-E_j) \tag{49}$$

The area A_j covered in the conical/angular zone $\delta_{j+1} - \delta_j$ is large for big targets and increases with its distance from the shell, and also includes, in general, portions of the target surface which do not fall under the given specified damage criteria. In such cases smaller areas around the vulnerable point can be choosen for which it is sufficient to determine the angular zone in which it falls besides being given the equation to the remaining boundaries of the area so that the solid angle $\int d\omega_j$ may be determined. If $f(\delta_{j,j+1})$ is known by using firing trial data in Eqn. (45) then E_j is known and thus $P_{k_i j}$ gets determined for the smaller vulnerable area.

If the vulnerable point situated in the area A_j requires higher optimum fragment energy than that envisaged or incorporated by the warhead designer for defeating the target, it is essential that the exposed area A_j in the conical/angular zone $\delta_{j+1} - \delta_j$ should register a finite number (greater than one) of 'killer' or lethal hits having higher optimum fragment energies for the defeat of the target or for inflicting the required specified category of damage. In such cases, it is necessary to distinguish fragment hits as the ones which register 'killer' or lethal hits and those which register 'non-killer' or 'simple' hits over the area A_j. This takes into account those fragments emitted in actual firing trails which belong to weight-groups that are neglected for having insufficient energy for lethal hits. In the following, the probability of lethal hits required for inflicting specified category of damag to the target out of a total number of r hits registered over an area is worked out and should be elaborated and interpreted under the concept of cumulative compound damage[2].

By Poisson law, the probability of exactly r hits out of a large number of hits is

$$P_{k_i j} \text{ hit } (r) = \frac{(E_j)^r}{r!} \exp(-E_j) \tag{50}$$

By Bernoulli's law, the probability of exactly ρ killer hits out of a group of r hits registered on the area is

$$P_{k_i j} \text{ kill } (\rho) = {}^rC_\rho \, p^\rho \, q^{r-\rho} \tag{51}$$

The chances of a hit to be 'killer' or 'simple' are equally likely and so their probabilities are equal and each is 1/2. Thus

$$P_{k_i j} \text{ kill } (\rho) = {}^rC_\rho / 2^r \tag{52}$$

Instead, if we take

$$E_j = r.\ p,\ p + q = 1 \tag{53}$$

for r large, p small and $r.p$ finite, we have

$$P_{k_i j} \text{ kill } (\rho) = {}^rC_\rho \left(\frac{E_j}{r}\right)^\rho \left(1 - \frac{E_j}{r}\right)^{r-\rho} \tag{54}$$

Hence, the probability that the area will receive exactly r hits (first event happens) and register exactly ρ killer hits (second event now happens) out of the r hits, is given by

$$P_{k_i j} \, H/K\,(\rho) = \frac{(E_j)^r}{r!} \exp(-E_j).\ {}^rC_\rho \, p^\rho \, q^{r-\rho} \tag{55}$$

where the probability factor for the second event is given by Eqn. (52) or (54)

Also, probability of exactly ρ killer hits out of atleast r hits is given by

$$P_{k_i j} \, H/K\,(\rho, \geqslant r) = 1 - \sum_{r=0}^{r-1} \frac{(E_j)^r}{r!} \exp(-E_j).\ {}^rC_\rho \, p^\rho \, q^{r-\rho} \tag{56}$$

It may be seen that for $\rho = r$, Eqn. (56) reduces to Eqn (49).

Calculations of the above different probabilities of kill for various numerical values of ρ and r of hits on various vulnerable part areas A_j of the target surface and their comparision will throw light on the extent to which the fragments/projectiles emanating in various conical/angular zones are effective. Thereafter, notching or preforming can suitably be redesigned/modified for the desired effectiveness of fragments emanating in various angular zones. In fact, this provides a method for analysing fragmentation data for the evaluation of warhead effectiveness.

8. Discussion and Plane Surface of a Rectangular Parallelopiped Type Target

The equations and various expressions obtained in the previous sections have been discussed at their places of derivation.

Charts of presented areas of targets for antipersonnel and antimaterial purposes can be computed for various positions and orientation of the shell and the target in space from the surface integral expression

$$A_p = \int_A \cos \Delta . \, dA \tag{57}$$

where cos Δ is given by Eqn. (35) and changing the surface integral to double integral and A is the exposed surface of the target to the fragments of the exploding shell.

On using Eqn (34) in Eqn. (57), the presented area can also be expressed by the surface integral

$$A_p = \int_A R^2 \, d\omega \tag{58}$$

where $d\omega$ and R^2 are given by relations from Eqn. (36) through Eqn. (43). It would be appropriate at this juncture to point out that the elementary area dA (or dA_j) or elementary solid angle $d\omega$ (or $d\omega_{j,j+1}$) belongs to the surface of the target represented by the general equation (39). When evaluating presented area or solid angle for a portion of such a surface the integration is to be carried out over points belonging to this portion of the target surface. This integration is achieved by carrying out integration over points contained in the projected area on the $x'y'$-coordinate plane by the given portion of the surface of the target. If dA'_j be the elementary projected area on the x' y'-coordinate plane by the elementary area dA_j of the target surface and if Δ_1 be the angle between the elementary planes dA_j and dA'_j, then

$$dA_j = \sec \Delta_1 . \, dA'_j = \left(\sqrt{l_j^2 + m_j^2 + n_j^2} \Big/ n_j \right) dx' \, dy' \tag{59}$$

and the integrations are to be carried out along the projected boundary cvrves. In the case of the plane target surface parallel to the x' y'-coordinate plane, we have $\Delta_1 = 0$.

The nature and utility of the equations and results obtained in the preceeding sections can be easily visualised by considering a simplified tactical situation in which the shell is moving with uniform velocity C_s along a straight line parallel to the x-axis in the zx-plane at distance d from the origin, and a rectangular parallelopiped (box) target of dimensions $2a_1$ x $2b_1$ x $2c_1$ moving with uniform velocity C_T along the x-axis. The problem is further simplified by taking the shell axis parallel to the x-axis and the normal to the faces $2a_1$ x $2b_1$ parallel to the z-axis (Fig. 3).

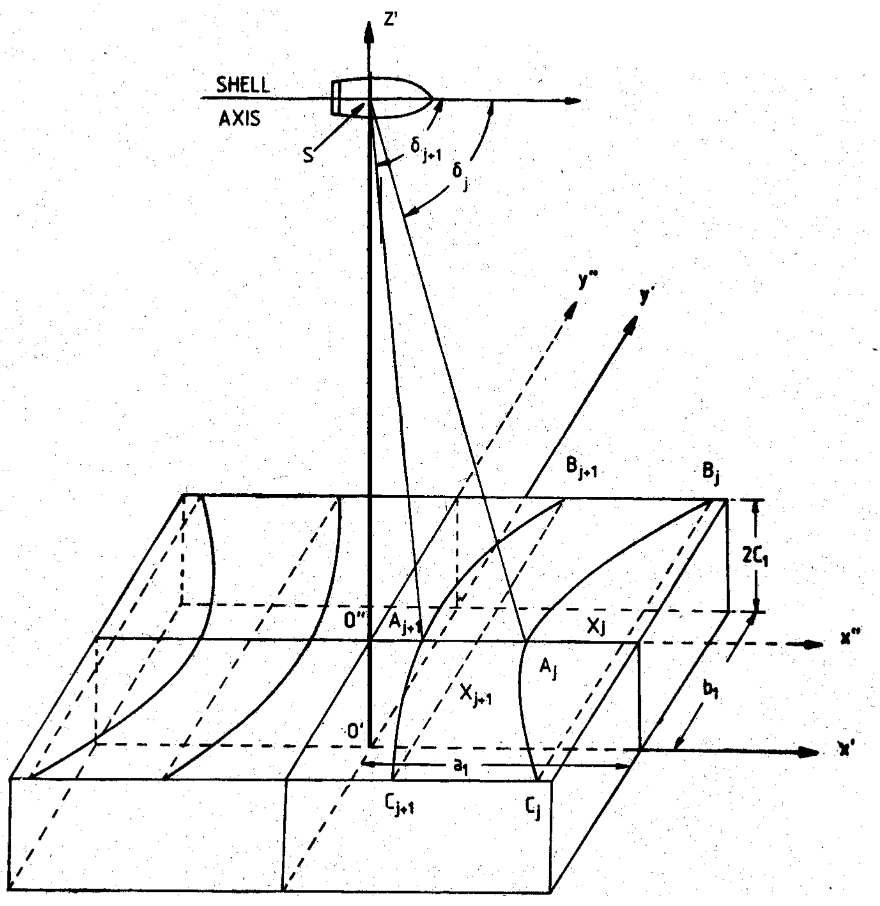

Figure 3. Hyperbolas B_j A_j C_j and B_{j+1} A_{j+1}, C_{j+1} formed by the interaction of right circular cones of semi-verticle angles δ_j and δ_{j+1} with the upper surface to the box target.

The parametric equations of the shell and the upper face $2a_1$ x 26, of the target referred to a fixed coordinate system can thus be written respectively as

$$x_s\,(t) = -kC_s + C_s\,t\,,\ y_s\,(t) = 0,\ z,(t) = d \tag{60}$$

and

$$x_0\,(t) = \mathrm{CT}\ t,\ \ y_0\,(t) = 0,\ \ z_0\,(t) = 0 \tag{61}$$

where k is a constant.

From Eqn. (60), we obtain

$$x_{os} = -\,kC_s,\ y_{os} = 0,\ z_{os} = d \tag{62}$$

$$x_1(t) = C_s\, t,\ y_1(t) = 0,\ z_1(t) = 0 \tag{63}$$

In practice, the trajectory of a shell is not a straight line but can be assumed to be so for small durations/distances by taking the value of k to be small.

For given values of U, V and λ (= 0 for shell axis tangential to the trajectory) at the time of burst, Eqn. (33) gives a family of cones for different values of the parameter δ_0, and therefore, for different values of the parameter θ due to the relation (21). Thus, from Eqns. (33). (61), (62) and (63) the equation to a family of moving cones having constant relative velocity $C_s - c_T$ along x'-axis, and formed around the shell axis with direction cosines $l_s = 1$, $m_s = n_s = 0$ is given by

$$[x' + \{kC_s - (C_s - C_T)\, t\}]^2 \sin^2 \delta_j - y'^2 \cos^2 \delta_j - (z'-d)^2 \cos^2 \delta_j = 0 \tag{64}$$

where $\delta_0 = \delta_j$ for the j-th cone.

Hence, the equation to the family of curves formed by the intersection of the above cones by the upper surfaces of the box-target are obtained by putting in Eqn. (64)

$$z' = 2c_1$$

A simble transformation shows that Eqns. (64) & (65) represent a family of hyperbolas on the upper surface Eqn. (65) of the box target for different values of δ_j

In order to find the solid angle subtended at S by the exposed are A, between the j-th hyperbola $B_j\ A_j\ C_j$ and $(j+1)$-th hyperbola $B_{j+1}\ A_{j+1}\ C_{j+1}$ corresponding to semi-vertical angles δ_j and δ_{j+1} of the respective cones, let us assumme that the solid angles subtended at S by the areas $A_j\, A_{j+1}\, B_{j+1}\, B_j$, $X_j\, B_{j+1}\, B_j\, X_j$, $X_{j+1}\, A_{j+1}\, B_{j+1}$ and $X_j\ A_j\ B_j$ be denoted by respectively $(1/2)\ \omega_{j,j+1}$, $(1/2)\ \omega_{\text{rect}}$ and $(1/2)\ \omega_j$. Thus, we have

$$\omega_{j,j+1} = w_{\text{rect}} + w_{j+1} - \omega_j = \int d\omega_{\text{rect}} + \int d\omega_{j+1} - \int d\omega_j \tag{66}$$

where $d\omega s$ are given by Eqn (46) and the integrations are to be carried out along curves bounding the exposed areas of the target configuration inside the solid angles.

From Eqn. (65) of the upper surface of the target, direction cosines of its normal are given by $l_j = m_j = 0$, $n_j = 1$. Hence, from Eqn. (46)

$$\omega_j = \int d\omega_j = \int_{x_1'}^{x_2'} \int_{y_1'(x)}^{y_2'(x')} \frac{(d - 2C_1)\, dy'\, dx'}{[\{-kC_s + (C_s - C_T)\, t - x'\}^2 + y'^2 + (d-2C_1)^2]^{3/2}}$$

$$= 2\left[\tan^{-1}\left(\frac{b_1}{D}\cos\delta_j\right) - \cos\delta_j \tan^{-1}\left(\frac{b_1}{D}\right)\right] \tag{67}$$

where

$$D = d - 2c_1 \tag{68}$$

and y_1' and y_2' are to be substituted from the equations of the bounding hyperbolas in terms of x', and x_1' and x_2' denote that the sub-target surface covered by the conical/ angular zone $\delta_{j+1} - \delta_j$ extends from $x = x_1'$ to x_2'.

Similarly,

$$\omega_{j+1} = 2\left[\tan^{-1}\left(\frac{b_1}{D}\cos\delta_{j+1}\right) - \cos\delta_{j+1}\tan^{-1}\left(\frac{b_1}{D}\right)\right] \tag{69}$$

Again, from Eqn. (46)

$$\omega_{\text{rect}} = \int_{x_3'}^{x_4'}\int_{y_3'(x')}^{y_4'(x')} \frac{(d-2C_1)\,dy'\,dx'}{[\{-kC_s+(C_s-C_T)\,t-x'\}^2+y'^2+(d-2C_1)^2]^{3/2}} \tag{70}$$

$$= 2\left[\tan^{-1}\left(\frac{b_1}{D}\cos\delta_j\right) - \tan^{-1}\left(\frac{b_1}{D}\cos\delta_{j+1}\right)\right] \tag{71}$$

Thus, on using Eqns. (67), (69) and (71) in Eqn. (66), we obtain the expression for the solid angle subtended at S by the area of the target surface covered in the conical zone $\delta_{j+1} - \delta_j$:

$$\omega_{j,j+1} = 2\,(\cos\delta_j - \cos\delta_{j+1})\tan^{-1}\left(\frac{b_1}{D}\right) \tag{72}$$

We know that the solid angle $\Omega_{j,j+1}$ (say) between the δ_j and δ_{j+1} cones is given by

$$\Omega_{j,j+1} = 2\pi\,(\cos\delta_j - \cos\delta_{j+1}) \tag{73}$$

The ratio $\Omega_{j,j+1}/\omega_{j,j+1}$ is called correction factor and is used to convert data taken in rectangular layouts constructed for static firing trials for determining spatial distribution of fragments

$$CF = \pi/\tan^{-1}\left(\frac{b_1}{D}\right) \tag{74}$$

It must be evident at this stage that when the shell axis is not parallel to the plane surface of the target, that is, for a finite value of angle of descent other than 0" or 180°, we are again required to find the correction factor in order to determine the number of fragments in the conical zone $\delta_{j+1} - \delta_j$. This can also be found out by a method exactly similar to that described in this paper.

From Eqns. (44), (45) and (72), the proportion of expected number of hits, E_j^*, on an area of the given target surface lying inside the conical zone $\delta_{j+1}^* - \delta_j^*$ which itself is covered by the large conical zone $\delta_{j+1} - \delta_j$, is thus given by

$$E_j^* = \frac{\sigma_j \left(\cos \delta_j^* - \cos \delta_{j+1}^*\right)}{N\pi \left(\cos \delta_j - \cos \delta_{j+1}\right)} \tan^{-1}\left(\frac{b_1}{D}\right) \tag{75}$$

For the case when $\delta_j^* = \delta_j$ and $\delta_{j+1}^* = \delta_{j+1}$, the above gives

$$E_j = \frac{\sigma_j}{N\pi} \tan^{-1}\left(\frac{b_1}{D}\right) \tag{76}$$

For Poisson law of small probabilities, the probability of a single hit, P, say, out of σ_j hits on the exposed target area under consideration, is given by

$$P = \frac{1}{N\pi} \tan^{-1}\left(\frac{b_1}{D}\right) \quad (E_j = \sigma_j p) \tag{77}$$

Thus, from Eqn. (47), the probability of atleast one lethal hit on the target surface covered in the conical zone $\delta_{j+1} - \delta_j$ is given by

$$P_{k_i j}\ (r \geqslant 1) = 1 - exp\left\{- \frac{\sigma_j}{N\pi} \tan^{-1}\left(\frac{b_1}{D}\right)\right\} \tag{78}$$

It may be mentioned that for $D = 0$, the family of hyperbolas Eqn. (64) degenerates into a pair of straight lines, and, therefore, the expression involving D are true for

$$D \neq 0 \tag{79}$$

Further, due to the condition (17), Eqn. (74), for $z' = 2c_1$ represents a family of cones only when

$$\delta_j \neq 0, \ \pm \frac{\pi}{2}, \pm \pi \tag{80}$$

The results are, therefore, true when Eqns (79) and (80) are simultaneously obeyed.

In general, a target surface can be divided into areas covered by various conical/angular zones for which we know $f\ (\delta_{j,j+1})$ from Eqn. (45) and firing data. After calculating the solid angle subtended by each of such areas at the shell (or subtended by any vulnerable area having mathematically defined boundaries) of the target surface, we compute E_j by means of Eqn. (44), which can be used to obtain various probabilities $P_{k_i j}$ for assessing the damage capability of the warhead against specified target.

9. Conclusions

The equations and results obtained are applicable to stationary targets such as buildings, bridges, structures, etc.; targets moving on the ground such as tanks, AFVs, APCs, etc.; airborne targets such as aircrafts, rockets, missiles, etc; and space targets such, as satellites and space vehicles, provided parametric equations of trajectories of both the shell and the target are given.

References

1. AMC Pamphlet 706-160, Engineering Design Handbook Part-i, Nov. 1962, p 4-175.
2. AMC Pamphlet 706-160, Engineering Design Handbook Part II, p 8-68.