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# Large Amplitude Free Vibrations of Axisymmetric Orthotropic Circular Plate

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Abstract. Large amplitude free vibrations of an axisymmetric circular plate of orthotropic material have been investigated. Generalised field equations in the dynamic case for such a plate have been derived (in the von Karman sense) in terms of displacement components. Time-periods for linear and nonlinear vibrations have been compared and corresponding results for isotropy have been deduced. Some numerical results for the ratio of time-periods vs. non-dimensional amplitudes have been presented in tables.

## 1. Introduction

Thin plates of different shapes frequently occur in many structures and the study of the bending properties of plates is imperative to a design engineer. With the increased use of strong and light-weight structures, especially in aero-space engineering, and in the study of vibrations of machine parts, many problems of nonlinear vibrations naturally arise where the complementary stresses in the middle plane of the plate must be taken into account in deriving the governing field equations of the plate.

Extensive studies on the large amplitude (nonlinear) vibrations of elastic plates have been made by Berger's method[1]. Although Berger's method has got some advantages over von Karman's method for its decoupled form, but recently Nowinski & Ohnabe[2] have pointed out certain inaccuracies in Berger's equations and concluded that these equations lead to meaningless results for movable edge conditions. Therefore von Karman's method should be resorted to for obtaining more accurate results until some alternative theory is set forth.

In this paper, classical field equations in terms of displacement components (in the von Karman sense) have been derived for an axisymmetric orthotropic circular plate.

The solution of these equations have been obtained for both clamped immovable and movable edge conditions. For both the cases relative time-periods for linear and nonlinear vibrations have been obtained and corresponding results for isotropy have been deduced. Some numerical results have been presented in the form of tables.

## 2. Derivation of Field Equations

We consider an axisymmetric orthotropic circular plate of thickness $h$ and radius $a$. The co-ordinate system is chosen such that the middle plane of the plate coincides with the $r$-$\phi$ plane, the origin of the co-ordinate system being the centre of the plate with the z-axis upwards.

For a thin plate the stress-strain relation can be expressed in terms of the matrix equation[3]

$$\begin{pmatrix} \tau_{rr} \\ \tau_{\phi\phi} \\ \tau_{rz} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ 2\, e_{12} \end{pmatrix} \tag{1}$$

where $\tau_{rr}$, $\tau_{\phi\phi}$ and $\tau_{rz}$ are the in-plane stresses, $C_{ij}$ are elastic constants for orthotropic material and $e_{ij}$ are the strain components given by the relations

$$e_{11} = u_{,r} + \frac{1}{2} w_{,r}^2 - z\, w_{,rr} \tag{2}$$

$$e_{22} = \frac{u}{r} - z\, w_{,r} \tag{3}$$

$$e_{12} = 0 \tag{4}$$

where $u\,(r, z)$ and $w\,(r, z)$ are the in-plane and transverse displacements respectively.

The stress resultants $N_{rr}$ and $N_{\phi\phi}$ and also the stress couples $M_{rr}$ and $M_{\phi\phi}$ are given by the following relations

$$N_{rr} = \int_{-h/2}^{h/2} \tau_{rr}\, dz = C_{11}\, h \left( u_{,r} + \frac{1}{2} w_{,r}^2 \right) + C_{12}\, hu/r \tag{5}$$

$$N_{\phi\phi} = \int_{-h/2}^{h/2} \tau_{\phi\phi}\, dz = C_{12}\, h \left( u_{,r} + \frac{1}{2} w_{,r}^2 \right) + C_{22}\, hu/r \tag{6}$$

$$M_{rr} = \int_{-h/2}^{h/2} z\, \tau_{rr}\, dz = -C_{11} h^3 (w_{,rr})/12 - C_{12} h^3 (w_{,r})/12r \tag{7}$$

$$M_{\phi\phi} = \int_{-h/2}^{h/2} z\, \tau_{\phi\phi}\, dz = -C_{12} h^3 (w_{,rr})/12 - C_{22} h^3 (w_{,r})/12r \tag{8}$$

The equations of equilibrium of the nonlinear theory for the axisymmetric case are given by

$$-\frac{d}{dr}(rN_{rr}) + N_{\phi\phi} = 0 \tag{9}$$

$$-\frac{d}{dr}\left(r \frac{dw}{dr} N_{rr}\right) - \frac{d}{dr}(r\,Q) = q\,r - \rho h\, w_{,tt} \tag{10}$$

$$-\frac{d}{dr}(r\, M_{rr}) + M_{\phi\phi} + Qr = 0 \tag{11}$$

Considering Eqns. (5), (6) and (9), one gets

$$r^2 u_{,rr} + ru_{,r} - C_{22}\, u/C_{11} = 1/2\ (C_{12}/C_{11} - 1)\ rw_{,r}^2$$
$$- r^2 w_{,r} w_{,rr} \tag{12}$$

Considering Eqns. (7), (8), (10) and (11), one gets

$$\frac{C_{11} h^3}{12}\left(w_{,rrrr} + \frac{2}{r} w_{,rrr}\right) - \frac{C_{22} h^3}{12}\left(\frac{1}{r^2} w_{,rr} - \frac{w_{,r}}{r^3}\right)$$
$$+ \rho h\, w_{,tt} = q + \frac{1}{r}\frac{d}{dr}\left(r \frac{dw}{dr} N_{rr}\right) \tag{13}$$

Equations (12) and (13) form the basic governing equations for the dynamic analysis of orthotropic circular plates.

### 3. Method of Solution

**For** free vibrations, we take $q = 0$ and express the deflection w $(r, t)$ in the separable form

$$w(r, t) = A\left[1 + \sum_{i=2,4,\ldots}^{\infty} A_i (r/a)^i\right] F(t) \tag{14}$$

$$\approx A\ [I + A_2\ (r/a)^2 + A_4\ (r/a)^4]\ F(t) \tag{15}$$

where A is the maximum deflection at the centre of the plate and the constants $A_2$ and $A_4$ are determined from the boundary conditions.

Considering Eqns. ( 12) and (15), one gets

$$r^2 u_{,rr} + r\, u_{,r} - \frac{C_{22}}{C_{11}} u = C_1 r^3 + C_2 r^5 + C_3 r^7 \tag{16}$$

where

$$C_1 = \frac{2}{a^4} A^2 F^2(t) A_2^2 (C_{12}/C_{11} - 3) \tag{17}$$

$$C_2 = \frac{8}{a^6} A^2 F^2(t) A_2^2 A_4 (C_{12}/C_{11} - 5) \tag{18}$$

$$C_3 = \frac{8}{a^8} A^2 F^2(t) A_4^2 (C_{12}/C_{11} - 7) \tag{19}$$

The general solution of Eqn. (16), finite,at the origin, is given by

$$u(r, t) = C_c r^k + \frac{C_1 r^3}{9 - k^2} + \frac{C_2 r^5}{25 - k^2} + \frac{C_5 r^7}{49 - k^2} \tag{20}$$

where $C_0$ is the constant of integration to be determined from the inplane boundary conditions for immovable and movable edges and

$$k^2 = C_{22}/C_{11}.$$

We now determine the constant of integration $C_0$. For clamped immovable edges of the plate, we have $u = 0$ at $r = a$ leading to

$$C_0 = -\left[\frac{2A_2^2 (C_{12}/C_{11} - 3)}{9 - k^2} + \frac{8A_2 A_4 (C_{12}/C_{11} - 5)}{25 - k^2} + \frac{8A_4^2 (C_{12}/C_{11} - 7)}{49 - k^2}\right] (a^{-k-1}) A^2 F^2(t) \tag{21}$$

For movable edges of a plate, we have $N_{rr} = 0$ at $r = a$ and one gets

$$C_0 = -\frac{a^{-k-1}}{k + C_{12}/C_{11}} \left[\frac{2A_2^2 \{(C_{12}/C_{11})^2 - 9\}}{25 - k^2} + \frac{8A_2A_4\{(C_{12}/C_{11})^2 - 25\}}{25 - k^2} + \frac{8A_4^2\{(C_{12}/C_{11})^2 - 49\}}{49 - k^2} + 2(A_2 - 2A_4)\right] A^2 F^2(t) \tag{22}$$

Substituting the expressions for w and $N_{rr}$ given by Eqns. (14) and (5) respectively into Eqn.(13) and applying Galerkin procedure one get, after a lengthy but simple evaluation of integrals, the nonlinear time differential equation in the form

$$\frac{d^2F(t)}{dt^2} + \alpha F(t) + \beta F^3(t) = 0 \tag{23}$$

where $\alpha$ and $\beta$ are positive constants given by lengthy expressions which have been omitted in the paper.

## 4. Boundary Conditions-Clamped Plate

For a plate clamped along the boundary

$$w = w_{,r} = 0 \text{ at } r = a \tag{24}$$

and considering Eqn. (15), one gets $A_2 = -2$, $A_1 = 1$. Accordingly, for both movable and immovable edges of a plate

$$\alpha = \frac{10\, C_{11}\, h^3}{9a^4\, \rho h}\left(9 - C_{22}/C_{11}\right) \tag{25}$$

and for immovable edges of a plate, one gets

$$\frac{10\, C_{11}\, A^2}{a^4 \rho}\Bigg[\Bigg\{-\frac{32\,(C_{12}/C_{11} - 3)}{9 - k^2} + \frac{64\,(C_{12}/C_{11} - 5)}{25 - k^2}$$

$$- \frac{32\,(C_{12}/C_{11} - 7)}{49 - k^2}\Bigg\}\left(\frac{1}{k+1} - \frac{2}{k+3} + \frac{1}{k+5}\right)$$

$$\times (k + C_{12}/C_{11})(k + 1)$$

$$+ \frac{16\{(C_{12}/C_{11})^2 - 9\}}{3\,(9 - k^2)} - \frac{32\{(C_{12}/C_{11})^2 - 25\}}{5\,(25 - k^2)} + \frac{32\{(C_{12}/C_{11})^2 - 49\}}{15\,(49 - k^2)}$$

$$+ \Bigg\{\frac{32\,(C_{12}/C_{11} - 3)}{9 - k^2} - \frac{64\,(C_{12}/C_{11} - 5)}{25 - k^2} + \frac{32}{3}\frac{(C_{12}/C_{11} - 7)}{49 - k^2}\Bigg\}\cdot$$

$$\left(\frac{1}{k+3} - \frac{2}{k+5} + \frac{1}{k+7}\right)(k + 3)(k + C_{12}/C_{11})$$

$$- \frac{16}{5}\frac{\{(C_{12}/C_{11})^2 - 9\}}{9 - k^2}$$

$$+ \frac{64}{15}\frac{\{(C_{12}/C_{11})^2 - 25\}}{25 - k^2} - \frac{32}{21}\frac{\{(C_{12}/C_{11})^2 - 49\}}{49 - k^2} + \frac{64}{105}\Bigg] \tag{26}$$

For movable edge the value of the constant $\beta$ is given by

$$\beta = \frac{10\, C_{11}\, A^2}{a^4 \rho} \left[ \frac{-32\, \{(C_{12}/C_{11})^2 - 9\}}{(k + C_{12}/C_{11})\,(9 - k^2)} + \frac{64\, \{(C_{12}/C_{11})^2 - 25\}}{(k + C_{12}/C_{11})\,(25 - k^2)} \right.$$

$$- \frac{32\, \{(C_{12}/C_{11})^2 - 49\}}{(k + C_{12}/C_{11})\,(49 - k^2)} \left( \frac{1}{k+1} - \frac{2}{k+3} + \frac{1}{k+5} \right)$$

$$(k + 1) \left( k + \frac{C_{12}}{C_{11}} \right) + \frac{16\, \{(C_{12}/C_{11})^2 - 9\}}{3\,(9 - k^2)}$$

$$- \frac{32\, \{(C_{12}/C_{11})^2 - 25\}}{5\,(25 - k^2)} \quad \frac{\{(C_{12}/C_{11})^2 - 49\}}{15\,(49 - k^2)}$$

$$+ \left\{ \frac{32\, \{(C_{12}/C_{11})^2 - 9\}}{(k + C_{12}/C_{11})\,(9 - k^2)} - \frac{64\, \{(C_{12}/C_{11})^2 - 25\}}{(k + C_{12}/C_{11})\,(25 - k^2)} \right.$$

$$\left. + \frac{32\, \{(C_{12}/C_{11})^2 - 49\}}{(k + C_{12}/C_{11})\,(49 - k^2)} \right\},$$

$$\left( \frac{1}{k+3} - \frac{2}{k \quad - \quad d} \quad \frac{1}{k+7} \right) (k + 3)\, (k + C_{12}/C_{11})$$

$$- \frac{16\, \{(C_{12}/C_{11})^2 - 9\}}{5\,(9 - k^2)} + \frac{64\, \{(C_{12}/C_{11})^2 - 25\}}{15\,(25 - k^2)} - \frac{32\, \{(C_{12}/C_{11})^2 - 49\}}{21\,(49 - k^2)}$$

$$\left. + \frac{64}{105} \right] \tag{27}$$

For isotropy,

$$C_{11} = C_{22} = \frac{E}{1 - \nu^2}, \quad \frac{C_{12}}{C_{11}} = \nu$$

and

$$k^2 = \frac{C_{22}}{C_{11}} = 1 \tag{28}$$

In this case the values of a and $\beta$ reduce to

(i) For clamped immovable edges

$$\alpha = \frac{320\, D}{3a^4\, \rho h}, \quad \beta = \frac{40\, D\,(1 + \nu) \left( \frac{23}{21} - \frac{3}{7} \nu \right) \left( \frac{A}{h} \right)^2}{a^4 \rho\, h} \tag{29}$$

(ii) For clamped movable edges the value of a is the same as in Eqn. (29) and the value of $\beta$ is given by

$$\beta = \frac{120\, D\,(1 - \nu^2)}{7\, a^4\, \rho h} (A/h)^2 \tag{30}$$

These values of a and $\beta$ are in good agreement with those obtained by modified Berger's method[4].,

The solution of Eqn. (23) with the initial conditions

$$F(O)= 1, \frac{dF(0)}{dt} = 0 \tag{31}$$

has been obtained by Nash and Modeer[4] with the help of Jacobian elliptic functions and accordingly the ratio of the nonlinear and linear time-periods $T^*/T$ is given by

$$T^*/T = \frac{2\Theta}{\pi}/(1 + \beta/\alpha)^{1/2} \tag{32}$$

For isotropy one gets the above relation in the form

$$T^*/T \text{ (clamped immovable)} = \frac{2\Theta}{\pi}/\{1 + 0.47\,1249\,(A/h)^2\}^{1/2}$$

$$T^*/T \text{ (clamped movable)} \quad \frac{2\Theta}{\pi}/\{1 + 0.14625\,(A/h)^2\}^{1/2} \tag{33}$$

## 5. Other Boundary Conditions

Corresponding results for other types of boundary conditions such as for simply-supported plates, can be derived by considering values of $A_2$ and $A_4$ admissible for those boundary conditions. This has been omitted in this paper.

## 6. Numerical Results

Numerical results for the relative time-periods $T^*/T$ for both immovable and movable edges of the plate have been, exhibited in the form of tables considering two types of orthotropic materials (Table 1).

Table 1. Orthotropic material constants[5]

| | $E_{11}$ | $E_{22}$ | $\nu_1$ | $\nu_2$ |
|---|---|---|---|---|
| Orthotropy I | 1 x 10' | 0.5 x $10^5$ | 0.05 | 0.025 |
| Orthotropy II | 1 x 105 | 0.05 x 10" | 0.2 | 0.01 |
| Isotropy | 1 x 10" | 1 x $10^5$ | 0.3 | 0.3 |

Table 2. Variations of relative time-periods $T^*/T$ *vs* relative amplitudes for a circular plates with clamped immovable edges

$C_{22}/C_{11} = E_{22}/E_{11}$. $C_{12}/C_{11} = \nu_1 E_{22}/E_{11}$

| $A/h$ | $T^*/T$ (Orthotropy-I) | $T^*/T$ (Orthotropy-II) | $T^*/T$ (Isotropy) |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 0.4 | 0.9743 | 0.9867 | 0.9643 |
| 0.8 | 0.9078 | 0.9569 | 0.8765 |
| 1.2 | 0.82189 | 0.9102 | 0.7712 |
| 1.4 | 0.7756 | 0.8833 | 0.7210 |
| 1.6 | 0.7344 | 0.85499 | 0.6732 |
| 1.8 | 0.6932 | 0.82599 | 0.6291 |
| 2.0 | 0.6543 | 0.7968 | 0.5887 |

Table 3. Variations of relative amplitudes vs relative, time-periods for a circular plate with clamped movable edges.

| $A/h$ | $T^*/T$ (Orthotropy-I) | $T^*/T$ (Orthotropy-II) | $T^*/T$ (Isotropy) |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 0.2 | 0.9981 | 0.9997 | 0.9907 |
| 0.4 | 0.9925 | 0.9989 | 0.9850 |
| 0.6 | 0.9833 | 0.9975 | 0.9746 |
| 0.8 | 0.9709 | 0.9956 | 0.9562 |
| 1.0 | 0.9557 | 0.9932 | 0.9340 |
| 1.2 | 0.9380 | 0.9903 | 0.9083 |

## 7. Conclusion

From the results of Tables 2 and3, the well-known phenomenon of decrease of relative time-periods with increasing amplitudes ('Hardening' type of nonlinearity) is observed. The hardening effect is considerably less for a plate with movable edge than that for plates with immovable edge.

The advantage of the method is that the same time-differential equation (23) provides results for both movable and immovable edges. This is the advantage over Berger's method.


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## References


1. Berger, H. M., *J. Appl. Mech., ASME*, **22 (1955)**, 465-472.
2. Nowinski, J. L. & Ohnabe, H., *Int. J. Mech.* Se., **14 (1972)**, 165-170.
3. Nowinski, J. L., 'Theory of Thermoelasticity with Applications' (Sijthoff and Noordhoff International Publishers), Alphen Ann Den Run, The Netherlands, 1978. 432-446.
4. Nash; W. A. & Modeer, J. H., 'Certain approximate analysis of the nonlinear behaviour of Plates and shallow shells', Proc. Sym on the theory of thin shells, Delft, The Netherlands, August 23-28, 1959.
5. Nowinski, J. L., *AIAA Journal*, **1 (1963)**, 619-630.