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An Algorithm for Calculating Expected Production Function and other Similar Functions for a Flexible Manlifacturing System

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Abstract. The fact that for the case when the number of machines in each group is the same, the expected production function is a symmetric function of scaled workloads, is exploited to express this function in terms of the basic symmetrio functions of the loads. The same technique is suggested for calculating the probability of all the machines being found busy and the variance of the proportion of busy machines in the system.

1. Introduction

During the last decade, a new information dominated, computer controlled and almost completely automated manufacturing 'technology designed to efficiently manufacture more than one kind of part with minimal set-up time, has emerged. This technology is represented by Flexible Manufacturing Systems (FMS's) which is comprised of numerically controlled workstations with an automated flow of information, work-pieces and tools under computer control.

Defence Manufacturing, with the exception of ammunition products that are, typically manufactured in large quantities, has two important characteristics viz. relatively low quantity production and high variety, and high technology products with a high rate of technical changes. The inherent flexibility of quick change over to new type of part or of adopting the product design changes by simply some adjustment of software is a great asset in defence manufacturing. More than two hundred FMS's are already in operation in developed countries like Japan, U.S.A., U.S.S.R., U.K. and West Germany and it would not be long when these are installed in a country like

India. To fully exploit the capabilities of these expensive systems, it is very important to understand their behaviour.

One successful approach to get insight of the behaviour of FMS is through mathematical modelling. A mathematical model which has been found quite useful is that of a Closed Network of Queues (CNQ) which is based on the phenomenon of queue formation.at the various machine-groups by a fixed number of parts circulating within the system for various operations. The theory of queueing networks was developed in early sixties because of its own mathematical interest and later on applied to performance evaluation of computer networks and other fields. It has been further developed in late seventies and eighties on account of its application to flexible manufacturing systems.

The development of theory and applications of cyclic queues and closed network of queues has been recently reviewed by Koenigsberg’. He has mentioned the applications of these theories to such important and diverse areas as communication networks and tele-traffic, computer time-sharing and multiprogramming systems, maintenance and repair facilities, production and assembly lines, inspection operations, and urban transportation systems. All these applications are of great interest to defence scientists.

Solberg[2,3] developed the theory of CNQ models for FMS’s. Stecke[4] and Stecke and Solberg[5] formulated five production planning problems related to FMS operation. In particular, the grouping and loading problems were modelled as non-linear zero-one mixed integer programming problems[6]. Stecke and Morin[7] showed that for adequately’ buffered FMS’s in which each operation is assigned to only one machine, balancing of workloads maximizes the expected production function. They showed that even though the objective function is not concave, it is sufficiently well-behaved to ensure that the local maximum is a global maximum. The behaviour of various CNQ models was considered by Suri[8] who defined a qualitative property called monotonicity for such a system and showed that the monoticity ensures that the system is well behaved with respect to a number of parameters. Suri[9] also considered the robustness of the product-form probability distribution for a CNQ model. This robustness has been examined from the point of view of maximum-entropy principle.

An important application, of significant interest to defence, of the theory of CNQ models has been made recently by Mani and Sarma[10] to aircraft availability and spares management.

In view of the large number of significant applications of CNQ models, the computation consideration of these models are of great importance. The direct method of calculating the various measures of performance like Expected Production Function, the Variance Production Function, Mean Queue Lengths etc. becomes unmanageable when the number of machine-groups and/or the number of parts in the system is large. Buzen[11] gave an' algorithm which simplifies the calculations considerably.

In the present paper, an alternative algorithm based on the concept of symmetric functions has been given and various measures. of performance in terms of basic symmetric functions is expressed so that once the basic symmetric functions are computed, the measures of performance can be obtained in a straight forward manner.

The calculations of the basic symmetric function is itself quite simple, particularly if one uses the mathematically-oriented APL computer language for computation. Moreover the use of the symmetric function gives an additional insight into the structure of the various measures of performance.

2. Expected Production Function

Let $s_1, s_2, \ldots, s_M$ be the number of machines in the M machine-groups on which the work loads are $x_1, x_2, \ldots, x_M$, where these workloads are scaled so as to give

$$x_1 + x_2 + \ldots + x_M = s_1 + s_2 + \ldots + s_M = m \tag{1}$$

Here m represents the total number of machines in the system. If N is the total number of parts in the system, then the probability that there are $n_1, n_2, \ldots, n_M$ parts in the various groups either being processed or in waiting to be processed, is given by (Gordon and Newell[12]).

$$p(n_1, n_2, \ldots, n_M) = \frac{g_1(n_1)\, g_2(n_2) \ldots g_M(n_M)}{\sum\limits_{S(M,N)} g_1(n_1)\, g_2(n_2) \ldots g_M(n_M)}$$

$$= \frac{g_1(n_1)\, g_2(n_2) \ldots g_M(n_M)}{G(M, N, X)} \tag{2}$$

where S $(M,$ N) is the set of all non-negative integers

$n_1, n_2, \ldots, n_M$ satisfying

$$n_1 + n_2 + \ldots + n_M = N, \qquad n_1 \geqslant 0, n_2 \geqslant 0, \ldots, n_M \geqslant 0 \tag{3}$$

and

$$g_i(n_i) = x_i^{n_i} \text{ for single machine machine-groups} \tag{4}$$

$$g_i(n_i) = \begin{cases} \dfrac{x_i^{n_i}}{n_i!} & \text{if } n_i \leqslant s_i \\[2ex] \dfrac{x_i^{n_i}}{s_i!\, s_i^{n_i - s_i}} & \text{if } n_i > s_i \end{cases} \quad \text{for multiple machine machine-groups} \tag{5}$$

and

$$X = (x_1, x_2, \ldots, x_M)$$

The proportion of busy machines in the system is a random variable whose expected value is given by Solberg[2,3], Kapur & Kumar[13] and Stecke[4].

$$G(M, N-1, X)/G(M, N, X) \tag{6}$$

This function called the Expected Production Function (EPF) has to be evaluated. An efficient algorithm for evaluating the EPF was given by Buzen[11] and a computer programme CANQ based on this algorithm was given by Solberg[2,3]. In the present paper, another algorithm is given. Though this algorithm is not as efficient as Buzen's algorithm, yet its derivation throws some interesting light on the symmetrical nature of this function.

3. Generating Function for G $(M,$ N, X) for Single Machine Machine-Groups

It is easily verified that

$$\phi(t) = \sum_{N=0}^{\infty} t^N G(M, N, X) = \prod_{i=1}^{M} \left(\sum_{n_i=0}^{\infty} g_i(n_i)\, t^{n_i} \right) = \prod_{i=1}^{M} \phi_i(t)$$

$$= \prod_{i=1}^{M} \left[1 + \frac{x_i t}{1!} + \frac{x_i^2 t^2}{2!} + \ldots + \frac{x_i^{s_i} t^{s_i}}{s_i!} + \frac{x_i^{s_i+1}}{s_i!\, s_i} t^{s_i+1} \right.$$

$$\left. + \frac{x_i^{s_i+2}}{s_i!\, s_i^2} t^{s_i+2} + \ldots \right], \tag{8}$$

so that the R.H.S. may be called the generating function for G $(M, N,$ X), since G $(M,$ N, X) is the coefficient of t^N in the expansion of the R.H.S., in powers oft.

For single machine machine-group, we get

$$\phi(t) = \sum_{N=0}^{\infty} t^N \mathrm{G}(M, \mathrm{N}, \mathrm{X}) = \prod_{i=1}^{M} \left(1 + x_i t + x_i^2 t^2 + \ldots \right)$$

$$= \prod_{i=1}^{M} (1 - x_i t)^{-1} \equiv \left[\prod_{i=1}^{M} (1 - x_i t) \right]^{-1} \tag{6}$$

provided

$$t < \min\left[\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_M}\right] \tag{10}$$

Now we define M basic symmetric functions (BSF) by

$$\left.\begin{aligned} S_1 &= \sum_{i=1}^{M} x_i,\; S_2 = \sum_{\substack{j=1\\ j>i}}^{M}\sum_{i=1}^{M} x_i x_j,\; S_3 = \sum_{\substack{k=1\\ k>j>i}}^{M}\sum_{j=1}^{M}\sum_{i=1}^{M} x_i\, x_j\, x_k \\ &\ldots\;\ldots\ldots\ldots\; S_M = x_1 x_2 \ldots x_M \end{aligned}\right\} \tag{11}$$

so that S_r is the sum of $\binom{M}{r}$ terms, each term being the product of r distinct x_i 's.

From Eqns. (9) and (1 I)

$$\phi(t) = \sum_{N=0}^{\infty} t^N\, G(M, N, X) = (1 - S_1 t + S_2 t^2 - S_3 t^3 + \ldots + S_M (-t)^M)^{-1} \tag{12}$$

Equating the coefficients of various powers of t, we get

$$\left.\begin{aligned} &G(M, 0, X) = 1,\; G(M, 1, X) = S_1, \quad G(M, 2, X) = S_1^2 - S_2 \\ &G(M, 3, X) = S_1^3 - 2S_1 S_2 + S_3,\; G(M, 4, X) \\ &\qquad = S_1^4 - 3S_1^2 S_2 + 2S_1 S_3 + S_2^2 - S_4 \\ &G(M, 5, X) = S_1^5 - 4S_1^3 S_2 + 3S_1^2 S_3 + 3S_1 S_2^2 - 2S_1 S_4 - 2S_2 S_3 + S_5, \ldots \end{aligned}\right\} \tag{13}$$

In the same way, we can express G $(M$, N, $X)$ in terms of $S_1, S_2, \ldots, S_M$ for all values of N. It may be noticed that when N $\leqslant M$, we require only first N basic symmetric functions, but when N $> M$, we require all the M basic symmetric functions and we require only these. Also the number of terms in G (M, N, X) is equal to the number of partitions $p_M^{(N)}$ of N into at most M positive integers. It also appears that the sum of the coefficients of all the terms in G $(M$, N, $X)$ is zero. To see whether it is always true, we put $S_1 = S_2 = \ldots = S_M = 1$ in Eqn (10) to get

$$\phi(t) = (1 - t + t^2 + \ldots + (-1)^M t^M)^{-1} = \left[\frac{1 - (-t)^{M+1}}{1 + t}\right]^{-1}$$

$$= (1 + t)(1 + (-t)^{M+1} + (-t)^{2M+2} + \ldots), \tag{14}$$

So that the sum of the coefficients in G (M, N, X) vanishes except when

$$N=0,\ 1,\ M+1,\ M+2,\ 2M+2,\ 2M+3,$$

The sum of the coefficients

$$\left.\begin{array}{l} = 1 \text{ if } N = 2k\ (M+1),\ 2k\ (M+1)+1, \\ = -1 \text{ if } N = (2k+1)(M+1),\ (2k+1)(M+1)+1 \\ = 0 \text{ otherwise} \end{array}\right\}, \tag{15}$$

where

$$k = 0, 1, 2, 3, \ldots$$

Equation (12) can be written alternatively as

$$(1-S_1t+S_2t^2-S_3t^3+S_4t^4-\ldots)\,(1+G(M,1,X)\,t+G(M,2,X)\,t^2+\ldots) = 1 \tag{16}$$

Equating the coefficients of various powers of t on **both** sides, we get

$$\left.\begin{array}{l} G(M,1,X) - S_1 = 0 \\ G(M,2,X) - G(M,1,X)\,S_1 + S_2 = 0 \\ G(M,3,X) - G(M,2,X)\,S_1 + G(M,1,X)\,S_2 - S_3 = 0 \\ G(M,4,X) - G(M,3,X)\,S_1 + G(M,2,X)\,S_2 - G(M,1,X)\,S_3 + S_4 = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \tag{17}$$

From these, we can calculate G $(M,$ 1, X), G (M, 2. X).. . ..in succession. The number of additions and multiplications required for calculating the basic symmetric functions of the algorithm are given by the following table

Functions	S_1	S_2	S_3	$\ldots S_r$	$\ldots S_M$
Number of additions	M - 1	$\binom{M}{2}-1$	$\binom{M}{3}-1$	$\binom{M}{r}-1$	..o
Number of multiplications	0	$\binom{M}{2}$	$\binom{M}{3}$ (2)	$\binom{M}{r}$ $(r-1)$	$\ldots M-1)$

For calculating S,, $S_2,\ldots,S_M$, we require 2^M-M-1 additions and 2^{M-1} (M-2)+ 1 multiplications. In addition, for calculating $G(M,1,X)$, $G(M,2,X)\ldots G(M,N,X)$, we require $(N-1)$ N/2 additions and (N $-$ 1) $N/2$ multiplications. These numbers are larger than the numbers required for Buzen's algorithm. However we have here an alternative algorithm and this builds up the function G $(M,$ N, X) in terms of basic building blocks viz. the basic symmetric functions. Moreover these basic symmetric functions are also useful for calculating not only the EPF but also the variance of the proportion of busy machines or the probability of all the machines being found busy.

4. Calculation of Expected Production Function

Let $G(M, 0, X) = 1$, then

$$Pr(M, N, X) = \frac{G(M, N-1, X)}{G(M, N, X)}; \; N = 1, 2, 3, \ldots \tag{18}$$

then Eqn. (17) give

$$\left.\begin{array}{l} S_1 \, Pr(M, 1, X) - 1 = 0 \\ S_2 \, Pr(M, 2, X) \, Pr(M, 1, X) - S_1 \, Pr(M, 2, X) + 1 = 0 \\ S_3 \, Pr(M, 3, X) \, Pr(M, 2, X) \, Pr(M, 1, X) - S_2 \, Pr(M, 3, X) \\ \qquad Pr(M, 2, X) + S_1 \, Pr(M, 3, X) - 1 = 0 \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \end{array}\right\} \tag{19}$$

From these equations $Pr(M, 1, X)$, $Pr(M, 2, X)$...can be successively determined in terms of $S_1, S_2, \ldots, S_M$.

5. Generating Function for $G(M, N, X)$ for Multiple Machine Machine-Groups

In this case Eqn. (8) gives the generating function, but the function cannot be very, much simplified since the M infinite series on the R.H.S. of Eqn. (8) cannot be summed up in convenient closed forms.

Moreover when $s_1, s_2, \ldots, s_M$ are not all equal, the generating functions and therefore $G(M, N, X)$'s are not symmetrical functions of $x_1, x_2, \ldots, x_M$. As such the problem of expressing $G(M, N, X)$ in terms of $S_1, S_2, \ldots, S_M$ does not arise.

However when $s_1 = s_2 = \ldots = s_M = s$, the generating function is a symmetric function of $x_1, x_2, \ldots, x_M$ and as such it should be possible to express all $G(M, N, X)$'s in terms of $S_1, S_2, \ldots, S_M$. It is easily seen that

$$\left.\begin{array}{l} G(M, 1, X) = \sum_{i=1}^{M} x_i = s_1 \\[2ex] G(M, 2, X) = \dfrac{x_1^2 + x_2^2 + \ldots + x_M^2}{2!} + \dfrac{x_1 x_2 + x_1 x_3 + \ldots + x_{M-1} x_M}{1!} \\[2ex] \qquad = \dfrac{S_1^2 - 2S_2}{2} + \dfrac{S_2}{1} = \dfrac{1}{2} S_1^2 \\[2ex] G(M, 3, X) = \dfrac{\sum_{i=1}^{M} x_i^3}{3!} + \dfrac{\sum_{\substack{j=1 \\ j \neq i}}^{M} \sum_{i=1}^{M} x_i^2 x_j}{2!} + \dfrac{\sum_{k=1}^{M} \sum_{j=1}^{M} \sum_{\substack{i=1 \\ i \neq j \neq k}}^{M} x_i x_j x_k}{1!} = \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \end{array}\right\} \tag{20}$$

It is obvious that the numerators of all the terms on the R.H.S. are symmetric functions of $x_1, x_2, \ldots, x_M$ and can be expressed in terms of S,, $S_2, \ldots, S_M$.

The pattern given in Eqn. (20) continues up to G (M, s, X) and changes some after that. However all G (M, N, X)'s can be expressed in terms of $S_1, S_2, \ldots S_M$.

The difference from the single machine machine-groups case arises in the sense that in the latter case, the denominators in the R.H.S. would all be I!.

6. Convergence of the Generating Function Series

For single machine machine-groups, from Eqns. (9) and (10), the series? $t^B_{N=1}$ G $(M$, N, $X)$ converges if Eqn. (10) is satisfied and its sum is given by Eqn. (9). For the general case,

$$\underset{N\to\infty}{Lt} \frac{U_N}{U_{N+1}} = \underset{N\to\infty}{Lt} \frac{t^N G(M, N, X)}{t^{N+1} G(M, N+1, X)}$$

$$= \frac{1}{t} \underset{N\to\infty}{Lt} Pr(M, N+1, X) \tag{21}$$

t is however know that

$$\underset{N\to\infty}{Lt} Pr(M, N, X) = \text{Min}\left(\frac{s_1}{x_1}, \frac{s_2}{x_2}, \ldots, \frac{s_M}{x_M}\right) \tag{22}$$

As such the generating function series coverges if

$$\frac{1}{t} \text{Min}\left(\frac{s_1}{x_1}, \frac{s_2}{x_2}, \ldots, \frac{s_M}{x_M}\right) > 1$$

or

$$t < \text{Min}\left(\frac{s_1}{x_1}, \frac{s_2}{x_2}, \ldots, \frac{s_M}{x_M}\right), \tag{23}$$

which in the case $s_1 = s_2 = \ldots = s_M = 1$ reduces to Eqn. (10)

Alternatively from Eqns. (7) and (8)

$$\phi_i(t) = 1 + x_i t + \frac{x_i^2}{2!} t^2 + \ldots + \frac{x_i^{s_i} t^{s_i}}{s_i!}\left[+ \frac{x_i t}{s_i} + \frac{x_i^2 t^2}{s_i^2} + \ldots\right] \tag{24}$$

$$= 1 + x_i t + \frac{x_i^2 t^2}{2!} + \ldots + \frac{x_i^{s_i} t^{s_i}}{s_i!}\left(1 - \frac{x_i t}{s_i}\right)^{-1}, \tag{25}$$

provided

$$t < \frac{s_i}{x_i} \tag{26}$$

Thus $\phi_1(t), \phi_2(t), \ldots, \phi_n(t)$ represent, convergent series when Eqn. (23) is satisfied, in which case the generating function is given by

$$\phi(t) = \prod_{t-1}^{M}\left[1 + x_i t + \frac{x_i^2 t^2}{2!} + \ldots + \frac{x_i^{s_i} t^{s_i}}{s_i!}\left(1 - \frac{x_i t}{s_i}\right)^{-1}\right] \tag{27}$$

7. Some Relations Between G (M, N, X) and S,, $S_2, \ldots, S_M$

When $s_1 = s_2 = \ldots = s_M = s$, Eqn. (27) gives

$$[1 + G(M,1,X)\, t - G(M,2,X)\, t^2 + \ldots]\left[\left(1 - \frac{tx_1}{s}\right)\left(1 - \frac{tx_2}{s}\right)\ldots\left(1 - \frac{tx_M}{s}\right)\right]$$

$$= \prod_{i=1}^{M}\left[\left(1 + x_i t + \ldots + \frac{x_i^{s-1} t^{s-1}}{(s-1)!}\right)\left(1 - \frac{x_i t}{s}\right) + \frac{x_i^s t^s}{s!}\right]$$

or

$$[1 + G(M,1,X)\, t + G(M,2,X)\, t^2 + \ldots]$$

$$\left[1 - \frac{S_1 t}{s} + \frac{S_2 t^2}{s^2} - \frac{S_3 t^3}{s^3} + \ldots + \frac{x_i^{s-1} t^{s-1}}{s^M}\right]$$

$$= \prod_{i=1}^{M}\left[1 + x_i t + \frac{x_i^2 t^2}{2!} + \ldots + \frac{x_i^{s-1} t^{s-1}}{(s-1)!}\right.$$

$$\left. - \frac{x_i t}{s} - \frac{x_i^2 t^2}{s} - \ldots - \frac{x_i^{s-1} t^{s-1}}{(s-1)!\, s}\right]$$

$$= \prod_{i=1}^{M}\left[1 + k_1 x_i t + k_2 x_i^2 t^2 + \ldots + k_{s-1} x_i^{s-1} t^{s-1}\right] \tag{28}$$

$$= 1 + k_1 tS_1 + k_2 t^2 \sum_{i=1}^{M} x_i^2 + k_1^2 t^2 \sum_{j=1}^{M} \sum_{\substack{i=1 \\ j>i}}^{M} x_i x_j$$

$$+ k_3 t^3 \sum_{i=1}^{M} x_i^3 + k_1 k_2 t^3 \sum_{j=1}^{M} \sum_{i=1}^{M} x_i^2 x_j + \ldots\ldots, \tag{29}$$

where

$$k_1 = 1 - \frac{1}{s},\ k_2 = \frac{1}{2!} - \frac{1}{s} \ldots k_{s-1} = \frac{1}{(s-1)!} - \frac{1}{(s-2)!s} \tag{30}$$

All the coefficients are symmetric functions of $x_1, x_2, \ldots, x_M$ and can be expressed in terms of $S_1, S_2, \ldots, S_M$. Thus $G_1 (M, N, X)$ can be expressed in terms of $S_1, S_2, \ldots, S_M$. We can prepare tables expressing $G (M, N, X)$ in terms of $S_1, S_2, \ldots, S_M$ for each s and these tables can be stored in the computer memory.

8. Number of Possible States

Putting $x_1 = x_2 = \ldots = x_M = 1$ in Eqn. (9), we get

$$\sum_{N=0}^{\infty} t^N G (M, N, \vec{1}) = (1 - t)^{-M};\ \vec{1} = (1, 1, \ldots, 1) \tag{31}$$

so that

$$G (M, N, \vec{1}) = \frac{(-M)(-M-1)\ldots(-M-N+1)}{N!}(-1)^N$$
$$= {}^{N+M-1}C_N = {}^{M+N-1}C_{M-1} \tag{32}$$

But

$$G (M, N, 1) = \left[\sum_{S(M,N)} x_1^{n_1} x_2^{n_2} \ldots x_M^{n_M}\right|_{x_1 = x_2 = \ldots = x_M = 1}$$

= Number of terms in G (M, N, X)

= Number of possible states

so that the number of possible states is ${}^{M+N-1}C_{M-1}$. This number is of course independent of the number of machines in each group.

9. Probability of all Machines being Busy

For single machine machine-groups, this is given by

$$H(M, N, X) = \frac{\sum\limits_{S'(M,N)} x_1^{n_1} x_2^{n_2} \ldots x_M^{n_M}}{\sum\limits_{S(M,N)} x_1^{n_1} x_2^{n_2} \ldots x_M^{n_M}}, \tag{33}$$

where S' (M, N) is the set of all integers satisfying

$$n_1 + n_2 + \ldots + n_M = N;\ n_1 \geqslant 1,\ n_2 \geqslant 1, \ldots, n_M \geqslant 1 \tag{34}$$

so that

$$H(M, N, X) = x_1 x_2 \ldots x_M \frac{\sum\limits_{S(M,N-X)} x_1^{n_1} x_2^{n_2} \ldots x_M^{n_M}}{\sum\limits_{S(M,N)} x_1^{n_1} x_2^{n_2} \ldots x_M^{n_M}}$$

$$= x_1 x_2 \ldots x_M \frac{G(M, N-M, X)}{G(M, N, X)} \tag{35}$$

$$= x_1 x_2 \ldots x_M \frac{G(M, N-M, X)\, G(M, N-M+1, X)}{G(M, N-M+1, X)\, G(M, N-M+2, X)}, \ldots, \frac{G(M, N-1, X)}{G(M, N, X)}$$

$$= x_1 x_2 \ldots x_M \Pr(M, N-M+1, X) \Pr(M, N-M+2, X), \ldots, \Pr(M, N, X) \tag{36}$$

Now each production function is maximum, when

$$x_1 = x_2 = \ldots = x_M = 1, \tag{37}$$

and in this case $x_1 x_2 \ldots x_M$ is **also** maximum. Thus the probability of all machines being busy is maximum when Eqn. (37) is satisfied.

For multiple machine machine-groups

$$H(M, N, X) = \frac{x_1^{s_1}}{s_1!} \frac{x_2^{s_2}}{s_2!} \cdots \frac{x_M^{s_M}}{s_M!} \frac{\sum\limits_{\bar{S}(M,N)} \left(\frac{x_1}{s_1}\right)^{n_1 - s_1} \left(\frac{x_2}{s_2}\right)^{n_2 - s_2} \cdots \left(\frac{x_M}{s_M}\right)^{n_M - s_M}}{G(M, N, X)}$$

$$= \frac{x_1^{s_1}}{s_2!} \frac{x_2^{s_2}}{s_2!} \cdots \frac{x_M^{s_M}\, \bar{G}\left(M, N-m, \frac{x_1}{s_1}, \frac{x_2}{s_2}, \ldots, \frac{x_M}{s_M}\right)}{S_M!\; G(M, N, x_1, x_2, \ldots, x_M)} \tag{38}$$

where $\bar{G}$ (M, N, X) corresponds to the single machine machine-groups case. When $s_1 = s_2 = \ldots = s_M = s$, H (M, N, X) is a symmetric function of $x_1, x_2, \ldots x_M$.

Thus when the numbers of machines in all the groups are the same, the probability of all the machines being found busy can be expressed in terms of $S_1, S_2, \ldots, S_M$.

10. Calculation of Variance

The variance of the proportion of busy machines is given by (Kapur & Kumar[18])

$$\frac{G(M, N-1, X)}{G(M, N, X)} \left[\frac{G(M, N-2, X)}{G(M, N-1, X)} - \frac{G(M, N-1, X)}{(M, N, X)} + \frac{1}{m} \right.$$

$$\left. - \frac{1}{m} \sum_{i=1}^{M} x_i \sum_{k=s_i}^{M-1} g_i(k) \frac{G_i(M-1, N-1-k, X)}{G(M, N-1, X)} \right] \tag{39}$$

If $s_1 = s_2 = \ldots = s_M = 1$, this gives

$$\frac{G(M, N-1, X)}{G(M, N, X)} \left[\frac{G'(M, N-2, X)}{G(M, N-1, X)} - \frac{G(M, N-1, X)}{G(M, N, X)} + \frac{1}{M} \right.$$

$$\left. - \frac{1}{M^2} \frac{G(M, N-2, X)}{G(M, N-1, X)} \sum_{i=1}^{M} x_i^2 \right] \tag{40}$$

$$Pr(M, N, X) \left[Pr(M, N-1, X) - Pr(M, N, X) + \frac{1}{M} \right.$$

$$\left. - \frac{1}{M^2} Pr(M, N-1, X) \left\{ S_1^2 - 2S_2 \right\} \right] \tag{41}$$

By using Eqn. (19), the Eqn. (41) can be expressed in terms of $S_1, S_2, \ldots, S_M$.

When $s_1 = s_2 = \ldots = s_M = s$, the Eqn. (39) is also a symmetric function of $x_1, x_2, \ldots, x_M$ and can be expressed in terms of $S_1, S_2, \ldots, S_M$.

11. Conclusions

From the above discussion, it appears that in the case when $s_1 = s_2 = \ldots = s_M = s$, all quantities of interest including EPF, probability of all machines being busy and the variance of proportion of the busy machines can be expressed in terms of the basic symmetric functions $S_1, S_2, \ldots, S_M$ and the calculations of all these quantities are relatively straight-forward once $S_1, S_2, \ldots, S_M$ have been calculated.

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