Def Sci J, Vol 36, No 1, January 1986, pp 63-70

Non-Slender Missile Geometries of Minimum Ballistic Factor via Calculus of Variations

MANAK SINGH & V. B. TAWAKLEY
Defence Science Centre, Delhi-110054

Received 6 May 1985; revised 13 September 1985

Abstract. The problem of determining the geometry of non-slender axisymmetric missile of minimum ballistic factor in hypersonic flow has been solved via calculus of variation under the assumption that the flow is Newtonian and the surface averaged skin friction coefficient is constant. The study has been made for conditions of given diameter and surface area. The results obtained have been compared with those obtained by the method of gradient technique.

Nomenclature

- D Drag force
- l Length of the missile
- q Free stream dynamic pressure
- S Surface area of the missile
- V Volume of the missile
- x Abscissa
- y Ordinate
- X Dimensionless abscissa
- Y Dimensionless ordinate
- y' Differentiation with respect to x i.e. dy/dx

1. Introduction

The problem of determining the slender missile shapes of minimum ballistic factor in hypersonic flow regime with supplementary conditions on the geometrical quantities of the missile viz. length, diameter, surface area and volume has been solved[1-4] via the calculus of variations. However, no such attempt has been made so far for calculating non-slender missiles. In this paper, the problem of determining non-slender shapes via variational calculus has been solved under the conditions that diameter d and surface area S of the body are known a priori. The results obtained have been compared with those obtained by the authors in an earlier paper[5] by using the method of gradient which is purely a numerical computing tool.

2. Formulation of the Problem

Considering the Newtonian flow theory and asssuming that the body is at zero angle of attack (Fig. 1), the pressure drag the surface area and the volume for a non-slender body are respectively given by

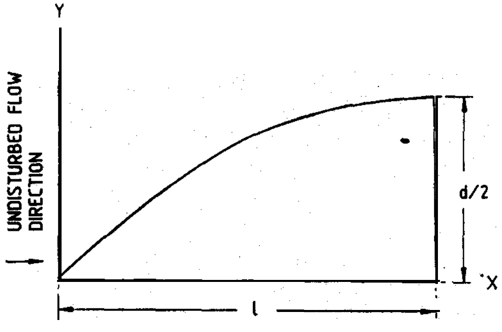

Figure 1. Coordnate systems.

$$\frac{D}{4\pi q} = \int_0^l \frac{y\, y'^3}{1 + y'^2}\, dx \tag{1}$$

$$S = 2\pi \int_0^l y\, (1 + y'^2)^{\frac{1}{2}}\, dx \tag{2}$$

$$V = \pi \int_0^l y^2\, dx \tag{3}$$

where q is the free stream dynamic pressure and I denote the length of the body.

Taking $X = \frac{x}{l}$, $Y = \frac{y}{l}$ as the dimensionless coordinates in the x and y directions respectively, we can write Eqns. (1) to (3) as follows :

$$D = 4\pi\, ql^2\, I_1$$

$$S = 2\pi\, l^2\, I_2$$

$$V = \pi\, l^3\, I_3$$

where

$$I_1 = \int_0^1 \frac{Y Y'^3}{1 + Y'^2}\, dx \tag{4}$$

$$I_2 = \int_0^1 Y (1 + Y'^2)^{\frac{1}{2}}\, dx \tag{5}$$

$$I_3 = \int_0^1 Y^2\, dx \tag{6}$$

We know that the ballistic factor of a missile is proportional to the ratio D/qV. Since the surface area and diameter are prescribed we can express ballistic factor as

$$I = \frac{32\pi}{S} \frac{I_2\, I_1^2}{I_3^2} \tag{7}$$

3. Solution of the Problem

From Eqn, (7) we observe that in general the problem of minimising the. ballistic factor is identical with that of minimising the functional

$$I = (I_1)^{\alpha_1} (I_2)^{\alpha_2} (I_3)^{\alpha_3}$$

where $\alpha_1 = 2$, $\alpha_2 = 1$, $\alpha_3 = -2$.

According to the theory given by Miele[6], the problem is governed by the following auxiliary function

$$F = \lambda_1 \frac{YY'^3}{1 + Y'^2} + \lambda_2\, Y\, (1 + Y'^2)^{\frac{1}{2}} + \lambda_3\, Y^2 \tag{8}$$

where λ_1, λ_2 and λ_3 are constant Lagrange multipliers given by

$$\lambda_1 = \frac{\alpha_1}{I_1},\ \lambda_2 = \frac{\alpha_2}{I_2},\ \lambda_3 = \frac{\alpha_3}{I_3} \tag{9}$$

Since the fundamental function given by Eqn. (8) does not contain the independent variable X explicitly, the Euler Equation of the problem must admit the following first integral:

$$- \frac{2\lambda_1 \; YY'^3}{(1 + Y'^2)^2} + \frac{\lambda_2 \, Y}{(1 + Y'^2)^{\frac{1}{2}}} + \lambda_3 Y^2 = c \tag{10}$$

where C is an integration constant.

Since the length is free the transversality condition leads to C = 0 and therefore the Eqn. (10) gives

$$\lambda \, Y = \frac{2Y'^3}{(1 + Y'^2)^2} - \frac{\mu}{(1 + Y'^2)^{\frac{1}{2}}} \tag{11}$$

where

$$\lambda = \frac{\lambda_3}{\lambda_1}, \; \mu = \frac{\lambda_2}{\lambda_1}$$

We will now obtain the solution of the Eqn. (11) in parametric form. For this we make the substitution

$$Y' = \tan t$$

and Eqn. (11) becomes

$$\lambda Y = 2 \sin^3 t \cos t - \mu \cos t \tag{12}$$

At the initial point Y = 0 and $t = t_0$ (unknown) and therefore, Eqn. (I 2) gives that

$$\mu = 2 \sin^3 t_0 \tag{13}$$

Also from the relation

$$dX = \frac{dY}{Y'} = \frac{1}{\lambda} [\sin 4t + \sin 2t + \mu \cos t] \, dt \tag{14}$$

We have on integration

$$X = - \frac{1}{\lambda} \left[\frac{\cos 4t}{4} + \frac{\cos 2t}{2} - \mu \sin t \right]_{t_0}^{t} \tag{15}$$

Since at the final point $Y = \frac{d}{2l}$, $t = t_f$ (both unknown), we obtain from Eqn. (I 2)

$$\lambda \frac{d}{2l} = 2 \sin^3 t_f \cos t_f - \mu \cos t_f \tag{16}$$

Also Eqn. (5) gives

$$\frac{d^2}{4l^2} = \frac{1}{\nu}\int_0^1 Y\,(1+Y'^2)^{\frac{1}{2}}\,dx \tag{17}$$

where

$$\nu = \frac{2s}{\pi d^2}$$

Combining the Eqns. (11), (14), (16) and (17), we obtain

$$(2\sin^3 t_f \cos t_f - \mu\cos t_f)^2 = \frac{1}{\nu}\int^{t_f}(2\sin^3 t - \mu)\,[2\sin t\,(3-4\sin^2 t)+\mu]\cos t\,dt$$

This equation can also be written as

$$4\nu\left(Z_f^3 - Z_0^3\right)^2\left(1 - Z_f^2\right) = \left[-\frac{16}{7}Z^7 + \frac{12}{5}Z^5 + \frac{5}{2}\mu Z^4 - 3\mu Z^2 - \mu^2 Z\right]_{Z_0}^{Z_f} \tag{18}$$

where

$$Z_0 = \sin t_0\,,\quad Z_f = \sin t_f$$

Again,

$$\mu = \frac{\lambda_2}{\lambda_1} = \frac{\frac{1}{2}\int_0^1 \frac{Y Y'^3}{1+Y'^2}\,dx}{\int_0^1 Y(1+Y'^2)^{\frac{1}{2}}\,dx}$$

Making use of Eqns. (12) and (14), this can be written as

$$4Z_0^3 = \frac{\left[-\frac{8}{5}Z^{10} + \frac{3}{2}Z^8 + \frac{10}{7}\mu Z^7 - \frac{6}{5}\mu Z^5 - \frac{\mu^2 Z^4}{4}\right]_{Z_0}^{Z_f}}{\left[-\frac{16}{7}Z^7 + \frac{12}{5}Z^5 + \frac{5}{2}\mu Z^4 - 3\mu Z^2 - \mu^2 Z\right]_{Z_0}^{Z_f}} \tag{19}$$

Equations (18) and (19) can be solved for the two unknown quantities Z_0 and Z_f for specified values of ν. Further since $X = 1$ when $t = t_f$, Eqn. (15) leads to

$$\lambda = -\left[\frac{\cos 4t}{4} + \frac{\cos 2t}{2} - \mu \sin t\right]_{t_0}^{t_f}$$

Thus having known the values of t_0 and t_f, the value of λ can be known from the above and the shape profiles can then be calculated from Eqns. (12) and (15). The shapes so obtained have been plotted in Fig. 2 for known values of ν. Also to calculate the ballistic factor, curves have been drawn for I_1/I_3 and I_2 vs. ν and are represented in Figs. 3 and 4 respectively.

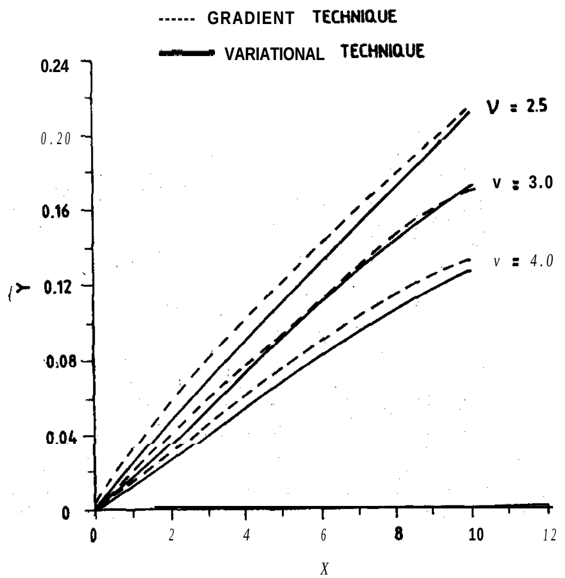

Figure 2. Optimum shapes of non-slender missiles.

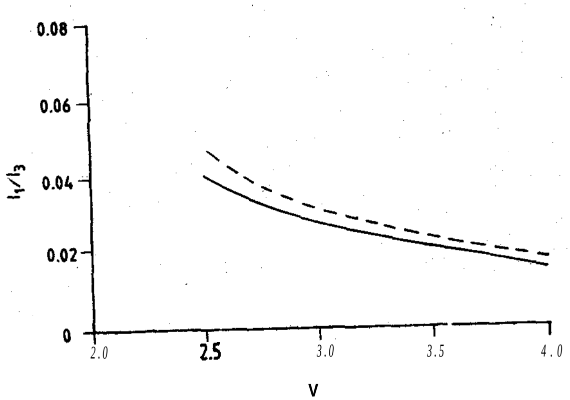

Figure 3. Product of ballistic factor and length for different values of ν.

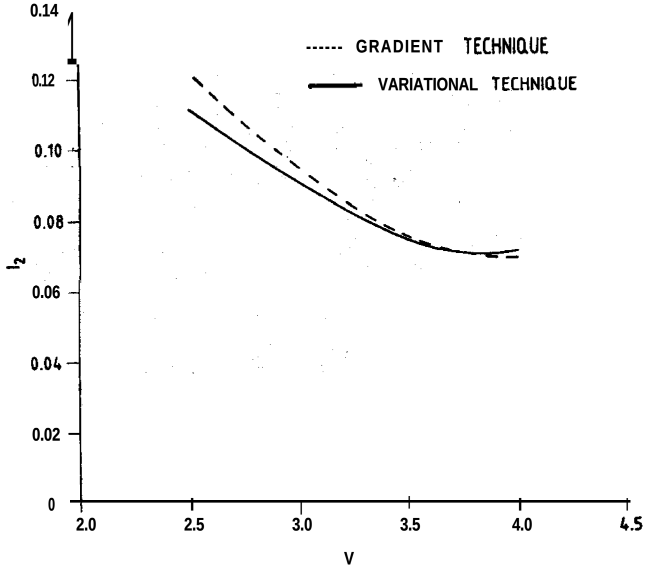

Figure 4. Graph for calculating optimum length of missile.

4. Conclusions

In an earlier paper, authors had reduced the problem of obtaining the configuration of missiles of minimum ballistic factor to a problem in optimum control and then solved the problem by utilising the gradient technique. Those results have been compared with the ones obtained above via the calculus of variations. In Fig. 2, the shapes obtained by the two methods for known values of v have been plotted while Fig. 3 and 4 respectively represent the values of I_1/I_3 and I_2 vs. v. Finally Fig. 5 depicts the values of $I' \left(\equiv \frac{D}{4qV}\sqrt{\frac{S}{2\pi}}\right)$ for known values of v.

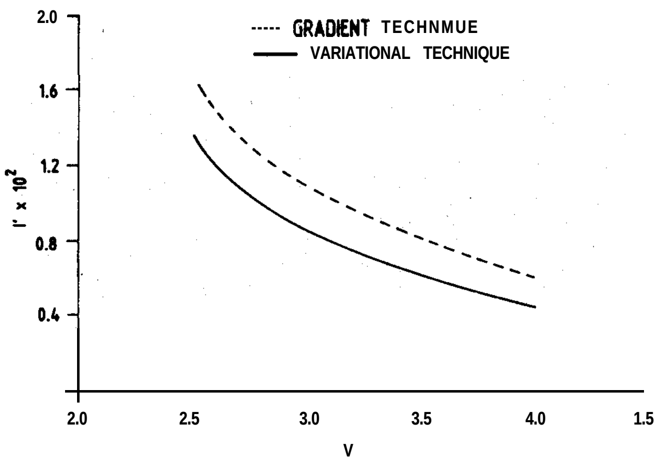

Figure 5. Ballistic factor for known value of v.

A comparison of the results by the gradient method and the calculus of variations indicates that the agreement is not far from accurate and any small discrepancy in the results by the two methods is due to the fact that firstly the gradient method is purely numerical based on the concept of local linearization around a nominal (non-optimal) solution involving iterative process while calculus of variation is analytic in nature. Secondly, first order gradient method usually show great improvements in the first few iterations but has poor convergence characteristics as optimal solution is approached.

References

1. Miele, A. & Huang, H. Y., *Aero Astronautics Rept.* 32, (Rice University, Houston Texas), April 1967; also *J. Optimization Theory & Applications*, 1 (1967), 151-164.
2. Tawakley, V. B. & Jain, S. C., *Astronautica Acta*, 18 (1973), 87-90.
3. Jain, S. C. & Tawakley, V. B., *Astronautica Acta* (Supplement), 18 (1974), 119-126.
4. Tawakley, V. B. & Jain, S. C., *Def. Sci, J.*, 29 (1979) 11-20.
5. Manak Singh & Tawakley, V. B., *Def. Sci. J.*, 31 (1981), 7-14.
6. Miele, A., *J. Optimization Theory and Applications*, No. 1 (1967), 70-82.