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Bayes Reliability Measures of Lognormal and Inverse Gaussian Distributions under ML-II ε-contaminated Class of Prior Distributions

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ABSTRACT

ML-II ε-contaminated class of priors are employed to study the sensitivity of Bayes reliability measures for a lognormal (LN) distribution and inverse Gaussian (IG) distribution to misspecification in the prior. The numerical illustrations suggest that reliability measures of both the distributions are not sensitive to moderate amount of misspecification in prior distributions belonging to the class of ML-II ε-contaminated.

Keywords: Bayes reliability, lognormal distribution, inverse Gaussian distribution, Bayesian methodology

1. INTRODUCTION

Often, in research and development of equipment for defence and for strategic purposes, a point of inflection is reached, that drives us in new directions suggesting the use of new materials, methods, and processes. There are always questions about how to ensure that the new equipment functions and performs as expected in the defence application for the length of time that is expected. What is the amount and method of testing that needs to be done to establish the expected reliability of the equipment in its final field applications? An evolutionary system design depends heavily on subjectivity held notions of reliability. This pushes us to reassess the methods previously used to test the reliability.

The advancement of computing power and the development of new computational methods has fostered the Bayesian methodology of reliability testing. The Bayesian methodology formulates prior probabilities to reflect all the existing subjective information available and construct a quantitative model to obtain reliability measures, taking care to incorporate the uncertainty inherent in the model assumption. The method utilises objective test data and investigator's subjective information to evaluate the reliability of new complex devices. Ke and Shen[1] propose an integrated Bayesian approach for reliability assessment during equipment development using the prior information; this approach can provide useful information for decision-making. Martz and Waller[2], and Blishke and Murthy[3] present excellent theory and applications of reliability analysis.

Robust Bayesian viewpoint assumes only that subjective information can be quantified in terms of a class of possible prior distributions. Any analysis, therefore, based on a single convenient prior distribution is questionable. A reasonable approach[4-7] is to consider a class of plausible priors that are in the neighbourhood of a specific assessed approximation to the 'true' prior and examine the robustness of the decision wrt this class of prior distributions.

Though the Markov Chain Monte Carlo (MCMC) method freed the analysts from using the conjugate prior for mathematical convenience but the problem still remains; how to eliminate the subjectivity involved in choosing a prior distribution?

The ε-contaminated class of prior distributions has attracted attention of a number of authors to model uncertainty in the prior distribution. Berger and Berliner[8] used type II maximum likelihood technique[9] to select a robust prior from ε-contaminated class of prior distributions having the form:

$$\Gamma = \left\{ \pi(\theta) = (1-\varepsilon)\,\pi_o + \varepsilon\, q\ ,\ q \in Q \right\}$$

Here, π_o is the true assessed prior and q, being a contamination, belongs to the class Q of all distributions. Q determines the allowed contaminations that are mixed with π_o, and $\varepsilon \in [0,1]$ reflects the amount of uncertainty in the 'true' prior π_o. ML-II technique would naturally select a prior with a large tail which would be robust against all plausible deviations. Sinha and Bansal[10] used ε-contaminated class of prior for the problem of optimisation of a regression nature in the decisive prediction framework.

The selection of the maximum likelihood type-II technique requires a robust prior π in the class Γ of priors, which maximises the marginal $m(\underset{\sim}{t} \mid \pi)$. For

$$\pi(\theta) = (1-\varepsilon)\pi_o(\theta) + \varepsilon q(\theta)\ ;\ q \in Q$$

the marginal of $\underset{\sim}{t}$

$$m(\underset{\sim}{t} \mid \pi) = (1-\varepsilon) m(\underset{\sim}{t} \mid \pi_o) + \varepsilon m(\underset{\sim}{t} \mid q)$$

can be maximised by maximising it over Q. Let the maximum

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of $m(\underset{\sim}{t} \mid q)$ be attained at unique $\hat{q} \in Q$. Thus, an estimated ML-II prior $\hat{\pi}(\theta)$ is given by

$$\hat{\pi}(\theta) = (1-\varepsilon)\pi_o(\theta) + \varepsilon\hat{q}(\theta)$$

The lognormal (LN) distribution is often useful in the analysis of economic, biological, and life-testing data. It can often be used to fit data that have large range of values. The lognormal distribution is commonly used for modelling asset prices, general reliability analysis, cycles-to-failure in fatigue, material strengths and loading variables in probabilistic design[11]. However, sometimes the lognormal distribution does not completely satisfy the fitting expectation in real situation; in such situations, the use of generalised form of lognormal distribution is suggested. Martín and Pérez[12] analysed a generalised form of lognormal distribution from a Bayesian point of view.

The two-parameter inverse Gaussian (IG) distribution, as a first passage time distribution in Brownian motion, found a variety of applications in the life testing, reliability and financial modelling problems. It has statistical properties analogous to normal distribution. Banerjee and Bhattacharyya[13] applied the IG distribution to consumer panel data on toothpaste purchase incidence for the assessment of consumer heterogeneity. Whitemore[14-15] discusses the potential applications of IG distribution in the management sciences and illustrates the advantages of IG distribution for right-skewed positive-valued responses and its applicability in stochastic model for many real settings. Aase[16] showed that IG distribution fits the economic indices, remarkably well in empirical investigations. Nadarajah and Kotz[17] gave the distribution of ratio of two economic indices each having IG distribution, for comparing the consumer price indices of six major economies.

Excellent monographs by Chhikara and Folks[18] and Seshadri[19] contain bibliographies and survey of the literature on IG distribution. Banerjee and Bhattacharyya[20] considered the normal distribution, truncated at zero, as a natural conjugate prior for the parameter θ of IG(θ, λ), while exploring the Bayesian results for IG distribution.

In the subsequent sections, the author have employed ML-II ε-contaminated class for the parameter θ of IG(θ, λ), shape parameter λ known, and $LN(\theta,\psi)$, ψ, known, to study sensitivity of Bayes reliability measures to misspecification in the prior distribution.

2. LOGNORMAL DISTRIBUTION

The probability density function (pdf) of lognormal distribution is expressed as

$$p(t \mid \theta,\psi) = \left(\frac{\psi}{2\pi}\right)^{1/2} t^{-1} \exp\left[-\frac{\psi}{2}\left(\ln(t)-\theta\right)\right], \quad t>0,\ -\infty<\theta<\infty,\ \psi>0 \tag{1}$$

where ψ is known and ln(t) is the natural log of t, Eqn (1) is designated by $LN(\theta,\psi)$.

Let $\underset{\sim}{t} = (t_1,\ldots,t_n)$ be n independent complete failure times from $LN(\theta,\psi)$. The likelihood function is given by

$$L(\theta \mid \underset{\sim}{t},\psi) = \left(\frac{\psi}{2\pi}\right)^{\frac{n}{2}} \prod_{i=1}^{n} t_i^{-1} \exp\left[-\frac{\psi v}{2} - \frac{n\psi}{2}(\theta-\bar{z})^2\right] \tag{2}$$

where $\nu = \sum_{i=1}^{n}\left(\ln(t_i)-\bar{z}\right)$ and $\bar{z} = \sum_{i=1}^{n}\ln(t_i)/n.$

The reliability for a time period of time t_o is

$$r(t_o;\theta,\psi) = P(T>t_o) = \int_{t_o}^{\infty} p(t\mid\theta,\psi)\mathrm{d}t = 1-\Phi\left(\sqrt{\psi}(\ln(t_o)-\theta)\right) \tag{3}$$

$\Phi(.)$ denotes standard normal cdf. Suppose θ has a prior distribution belonging to ML-II ε-contaminated class of priors. Following Berger and Berliner[8], we have $\pi_o(\theta)$ as $N(\mu,\tau)$ and $\hat{q}(\theta)$ as uniform $(\mu-\hat{a},\mu+\hat{a})$, $\hat{a}$ being the value of a which maximises

$$m(\underset{\sim}{t} \mid a) = \begin{cases} \dfrac{1}{2a}\displaystyle\int_{\mu-a}^{\mu+a} L(\theta \mid \underset{\sim}{t},\psi)d\theta & a>0 \\ L(\mu \mid \underset{\sim}{t},\psi) & a=0 \end{cases}$$

$m(\underset{\sim}{t} \mid a)$ is an upper bound on $m(\underset{\sim}{t} \mid q)$.

$$\begin{aligned} m(\underset{\sim}{t} \mid a) &= \left(\frac{\psi}{2\pi}\right)^{\frac{n}{2}} \prod_{i=1}^{n} t_i^{-1} e^{-\frac{\psi\nu}{2}} \sqrt{\frac{2\pi}{n\psi}} \frac{1}{2a} \int_{\mu-a}^{\mu+a} \sqrt{\frac{n\psi}{2\pi}} \exp\left[-\frac{n\psi}{2}(\theta-\bar{z})^2\right] d\theta \\ &= \frac{C}{2a}\left\{\Phi\left[\sqrt{n\psi}(\mu+a-\bar{z})\right] - \Phi\left[\sqrt{n\psi}(\mu-a-\bar{z})\right]\right\} \end{aligned} \tag{4}$$

where $C = \left(\frac{\psi}{2\pi}\right)^{\frac{n}{2}} \prod_{i=1}^{n} t_i^{-1} e^{-\frac{\psi\nu}{2}} \sqrt{\frac{2\pi}{n\psi}}$. On differentiating Eqn (4) wrt a, one has

$$\begin{aligned} \frac{d}{da} m(\underset{\sim}{t} \mid a) = &-\frac{C}{2a^2}\left\{\Phi\left[\sqrt{n\psi}(\mu+a-\bar{z})\right] - \Phi\left[\sqrt{n\psi}(\mu-a-\bar{z})\right]\right\} \\ &+ \frac{C\sqrt{n\psi}}{2a}\left\{\phi\left[\sqrt{n\psi}(\mu+a-\bar{z})\right] + \phi\left[\sqrt{n\psi}(\mu-a-\bar{z})\right]\right\} \end{aligned} \tag{5}$$

where $\phi(.)$ denotes standard normal pdf.

Now the authors substitute $\omega = \sqrt{n\psi}\,|\bar{z}-\mu|$ and $a^* = a\sqrt{n\psi}$ in Eqn (5) and equate to zero. The equation becomes

$$\Phi(a^*-\omega) - \Phi[-(a^*+\omega)] = a^*\left\{\phi(a^*-\omega) + \phi[-(a^*+\omega)]\right\}$$

which can be written as

$$a^* = \omega + \left\{-2\log\left[\sqrt{2\pi}\left(\frac{1}{a^*}\left[\Phi(a^*-\omega) - \Phi[-(a^*+\omega)]\right] - \phi[-(a^*+\omega)]\right)\right]\right\}^{\frac{1}{2}} \tag{6}$$

Solving Eqn (6) by standard fixed-point iteration, set $a^* = \omega$ on the right-hand side, which gives

$$\hat{a} = \begin{cases} 0 & \text{if } \omega \le 1.65 \\ \dfrac{a^*}{\sqrt{n\psi}} & \text{if } \omega > 1.65 \end{cases}$$

Following Berger and Sellke[21], the authors make $\hat{a}$ equal to zero when $\bar{t}$ is close to μ.

The usual Bayes point estimate $\hat{r}$, under quadratic loss function, is the posterior mean of $r(t_o;\theta,\psi)$

$$\hat{r} = \int_{\Theta} r(t_o;\theta,\psi)\pi(\theta \mid \underset{\sim}{t},\psi)d\theta \tag{7}$$

where posterior distribution $\pi(\theta \mid \underset{\sim}{t},\psi)$ of parameter θ wrt prior $\pi(\theta)$ is given by

$$\pi(\theta \mid \underset{\sim}{t},\psi) = \frac{L(\theta \mid \underset{\sim}{t},\psi)\pi(\theta)}{\lambda(\underset{\sim}{t})\int_{\Theta} L(\theta \mid \underset{\sim}{t},\psi)\pi_o(\theta)d\theta + (1-\lambda(\underset{\sim}{t}))\int_{\Theta} L(\theta \mid \underset{\sim}{t},\psi)q(\theta)d\theta}$$

$$= \frac{L(\theta \mid \underset{\sim}{t},\psi)\pi(\theta)}{\lambda(\underset{\sim}{t})m(\underset{\sim}{t} \mid \pi_o) + (1-\lambda(\underset{\sim}{t}))m(\underset{\sim}{t} \mid q)} = \lambda(\underset{\sim}{t})\pi_o(\theta \mid \underset{\sim}{t}) + (1-\lambda(\underset{\sim}{t}))q(\theta \mid \underset{\sim}{t}) \tag{8}$$

where

$$m(\underset{\sim}{t} \mid \pi_o) = C'e^{-\gamma} \;;\; C' = \left(\frac{\psi}{2\pi}\right)^{\frac{n}{2}} \prod_{i=1}^{n} t_i^{-1}\sqrt{\frac{\tau}{\tau'}} \;;\; \tau' = \tau + n\psi$$

$$m(\underset{\sim}{t} \mid q) = \frac{C}{2\hat{a}}\hat{\phi}_1 \;;\; \hat{\phi}_1 = \Phi\left[\sqrt{n\psi}\left(\mu+\hat{a}-\bar{z}\right)\right] - \Phi\left[\sqrt{n\psi}\left(\mu-\hat{a}-\bar{z}\right)\right],$$

$$\pi_o(\theta \mid \underset{\sim}{t}) = \frac{L(\theta \mid \underset{\sim}{t},\psi)\pi_o(\theta)}{m(\underset{\sim}{t} \mid \pi_o)} = \sqrt{\frac{\tau'}{2\pi}}\exp\left[-\frac{\tau'}{2}(\theta-\mu')^2\right] \;;\; \mu' = \frac{\tau\mu + n\psi\bar{z}}{\tau'}$$

$$q(\theta \mid \underset{\sim}{t}) = \frac{L(\theta \mid \underset{\sim}{t},\psi)q(\theta)}{m(\underset{\sim}{t} \mid q)} = \frac{1}{\hat{\phi}_1}\sqrt{\frac{n\psi}{2\pi}}\exp\left[-\frac{n\psi}{2}\left(\theta-\bar{z}\right)^2\right]$$

$$\lambda(\underset{\sim}{t}) = \left[1+\frac{\varepsilon m(\underset{\sim}{t} \mid q)}{(1-\varepsilon)m(\underset{\sim}{t} \mid \pi_o)}\right]^{-1} = \left[1+\frac{\varepsilon}{(1-\varepsilon)}\left(\frac{n\psi\tau}{2\pi\tau'}\right)^{-\frac{1}{2}}\frac{\hat{\phi}_1 e^{\gamma'}}{2\hat{a}}\right]^{-1}$$

$$\gamma = \gamma' + \frac{\psi v}{2} \text{ and } \gamma' = \frac{n\psi\tau}{2\tau'}\left(\mu-\bar{z}\right)^2$$

thus Eqn (7) becomes

$$\hat{r} = \int_{-\infty}^{\infty}\int_{t_o}^{\infty}\left(\frac{\psi}{2\pi}\right)^{\frac{1}{2}} t^{-1}\exp\left[-\frac{\psi}{2}\left(\ln(t)-\theta\right)^2\right]\pi_o(\theta \mid \underset{\sim}{t})dtd\theta$$

$$+\int_{\mu-\hat{a}}^{\mu+\hat{a}}\int_{t_o}^{\infty}\left(\frac{\psi}{2\pi}\right)^{\frac{1}{2}} t^{-1}\exp\left[-\frac{\psi}{2}\left(\ln(t)-\theta\right)^2\right]q(\theta \mid \underset{\sim}{t})dtd\theta$$

$$= \lambda(\underset{\sim}{t})\left\{1-\Phi\left[\sqrt{\tau''}\left(\ln(t_o)-\mu'\right)\right]\right\}$$

$$+\frac{(1-\lambda(\underset{\sim}{t}))}{\hat{\phi}_1}\left(\frac{n\psi}{2\pi(n+1)}\right)^{\frac{1}{2}}\int_{t_o}^{\infty} t^{-1}\exp\left[-\frac{n\psi}{2(n+1)}\left(\ln(t)-\bar{z}\right)^2\right]\phi(\underset{\sim}{t})dt \tag{9}$$

where

$$\phi(\underset{\sim}{t}) = \Phi\left[\sqrt{\psi(n+1)}\left(\mu+\hat{a}-\mu_1\right)\right] - \Phi\left[\sqrt{\psi(n+1)}\left(\mu-\hat{a}-\mu_1\right)\right] ;$$

$$\mu_1 = \frac{\ln(t)+n\bar{z}}{n+1},\; \tau'' = \frac{r\tau'}{r+\tau'}$$

Numerical integration was used to evaluate the incomplete integral in Eqn (9).

2.1 Lower One-sided Bayes Probability Interval Estimate

Reliability analysts are sometimes interested in 100 (1–α) per cent lower one-sided Bayes probability interval (LBPI) estimate r_* of $r(t_0)$ where α is chosen to be a small quantity. Bayesian estimate of $r(t_0)$ is easily constructed from the corresponding interval for θ as follows

$$P(\theta \le \theta_* \mid \underset{\sim}{t},\psi) = \alpha \tag{10}$$

Since $r(t_o;\theta,\psi)$ is a monotonically non-decreasing function of θ, one has the LBPI estimate of $r(t_0)$ as

$$P(\theta \le \theta_* \mid \underset{\sim}{t},\psi) = P\left(R(t_o) \le 1 - \Phi\left(\sqrt{\psi}(\ln(t_o)-\theta_*)\right) \mid \underset{\sim}{t},\psi\right) = \alpha$$

Thus 100(1–α) per cent LBPI estimate of $r(t_o)$ is given as

$$r_* = 1-\Phi\left(\sqrt{\psi}(\ln(t_o)-\theta_*)\right) \tag{11}$$

where θ_* is the 100(1–α) per cent LBPI estimate of θ and is evaluated as

$$\int_{\Theta}^{\theta_*}\pi(\theta \mid \underset{\sim}{t},\psi)d\theta = \alpha$$

$$\lambda(\underset{\sim}{t})\int_{-\infty}^{\theta_*}\pi_o(\theta \mid \underset{\sim}{t})d\theta + (1-\lambda(\underset{\sim}{t}))\int_{\mu-\hat{a}}^{\theta_*} q(\theta \mid \underset{\sim}{t})d\theta = \alpha$$

$$\lambda(\underset{\sim}{t})\Phi\left(\sqrt{\tau'}(\theta_*-\mu')\right) + \frac{(1-\lambda(\underset{\sim}{t}))}{\hat{\phi}_1}\Phi\left(\sqrt{n\psi}\left(\theta_*-\bar{z}\right)\right)$$

$$= \alpha + \frac{(1-\lambda(\underset{\sim}{t}))}{\hat{\phi}_1}\Phi\left(\sqrt{n\psi}\left(\mu-\hat{a}-\bar{z}\right)\right)$$

Author evaluated θ^* using Matlab for a given α and substituted in r_* to obtain the required LBPI for varying ε.

2.2 Reliable Life

The reliable life is the time t_R for which the reliability will be R. It may be considered as the time t_R for which 100R per cent of population will survive. The determination of t_R is the same as computing the 100(1–R)th percentile of the failure time distribution. For a $LN(\theta,\psi)$ population

$$t_R = \exp\left(\psi^{-1/2}\Phi^{-1}(1-R)+\theta\right)$$

For known ψ, t_R is the linear function of θ. The Bayes estimate of t_R, under quadratic loss function, is the posterior expected value of t_R

$$t_R = E^{\pi(\theta|\underset{\sim}{t},\psi)}\left[\exp\left(\psi^{-1/2}\Phi^{-1}(1-R)+\theta\right)\right]$$
$$=\exp\left(\psi^{-1/2}\Phi^{-1}(1-R)\right)\int_{\Theta} e^{\theta}\pi(\theta|\underset{\sim}{t},\psi)\,d\theta$$
$$=\lambda(\underset{\sim}{t})\int_{-\infty}^{\infty} e^{\theta}\pi_o(\theta|\underset{\sim}{t})\,d\theta+(1-\lambda(\underset{\sim}{t}))\int_{\mu-\hat{a}}^{\mu+\hat{a}} e^{\theta}q(\theta|\underset{\sim}{t})\,d\theta \qquad (12)$$
$$=\lambda(\underset{\sim}{t})\exp\left(\mu'+\frac{1}{2\tau'}\right)+\frac{(1-\lambda(\underset{\sim}{t}))}{\hat{\phi}_1}\exp\left(\bar{z}+\frac{1}{2n\psi}\right)\left[\Phi(h)-\Phi(g)\right]$$

where

$$h=\sqrt{n\psi}(\mu+\hat{a}-\bar{z})-\frac{1}{\sqrt{n\psi}} \quad \text{and} \quad g=\sqrt{n\psi}(\mu-\hat{a}-\bar{z})-\frac{1}{\sqrt{n\psi}}$$

3. INVERSE GAUSSIAN DISTRIBUTION

The probability density function (pdf) of IG distribution is expressed as

$$p(t\,|\,m,\lambda)=\left(\frac{\lambda}{2\pi}\right)^{1/2} t^{-3/2}\exp\left[-\frac{\lambda(t-m)^2}{2m^2t}\right], \qquad (13)$$

$t>0,\ \theta>0,\ \lambda>0$

where m and λ are the mean and shape parameters respectively.

Tweedie expressed Eqn (13) in terms of an alternative parameterisation, making $\theta=1/m$, as

$$p(t\,|\,\theta,\lambda)=\left(\frac{\lambda}{2\pi}\right)^{1/2} t^{-3/2}\exp\left[-\frac{\lambda t}{2}\left(\theta-\frac{1}{t}\right)^2\right], \qquad (14)$$

$t>0,\ \theta>0,\ \lambda>0$

Eqn (14) is denoted by IG(θ,λ).

Let $\underset{\sim}{t}=(t_1,\dots,t_n)$ be n independent complete failure times from IG(θ,λ) with mean $\theta=1/m$ and known shape parameter λ (>0). The likelihood function is given by

$$L(\theta\,|\,\underset{\sim}{t},\lambda)=\left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}}\prod_{i=1}^{n} t_i^{-\frac{3}{2}}\exp\left[-\frac{\lambda\nu}{2}-\frac{n\lambda\bar{t}}{2}\left(\theta-\frac{1}{\bar{t}}\right)^2\right] \qquad (15)$$

where $\nu=\sum_{i=1}^{n}\left(\frac{1}{t_i}-\frac{1}{\bar{t}}\right)$ and $\bar{t}=\sum_{i=1}^{n} t_i/n.$

The reliability for a time period of time t_o is

$$r(t_o;\theta,\lambda)=1-P(T\le t_o)$$
$$=\Phi\left(\sqrt{\frac{\lambda}{t_o}}(1-t_o\theta)\right)-e^{2\lambda\theta}\Phi\left(-\sqrt{\frac{\lambda}{t_o}}(1+t_o\theta)\right) \qquad (16)$$

Suppose θ has a prior distribution belonging to ML-II ε-contaminated class of priors, we have $\pi_o(\theta)$ as $N(\mu,\tau)$, truncated at zero, with pdf

$$\pi_o(\theta)=\frac{1}{G}\sqrt{\frac{\tau}{2\pi}}\exp\left[-\frac{\tau}{2}(\theta-\mu)^2\right];$$
$$G=\Phi(-p),\ p=-\mu\sqrt{\tau}$$

and $\hat{q}(\theta)$ as uniform $(\mu-\hat{a},\mu+\hat{a})$, $\hat{a}$ being the value of a which maximises

$$m(\underset{\sim}{t}\,|\,a)=\begin{cases}\dfrac{1}{2a}\displaystyle\int_{\mu-a}^{\mu+a} L(\theta\,|\,\underset{\sim}{t},\lambda)\,d\theta & a>0\\ L(\mu\,|\,\underset{\sim}{t},\lambda) & a=0\end{cases}$$

$m(\underset{\sim}{t}\,|\,a)$ is an upper bound on $m(\underset{\sim}{t}\,|\,q)$.

$$m(\underset{\sim}{t}\,|\,a)=\left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}}\prod_{i=1}^{n} t_i^{-\frac{3}{2}}e^{-\frac{\lambda\nu}{2}}\sqrt{\frac{2\pi}{n\lambda\bar{t}}}\frac{1}{2a}\int_{\mu-a}^{\mu+a}\sqrt{\frac{n\lambda\bar{t}}{2\pi}}\exp\left[-\frac{n\lambda\bar{t}}{2}\left(\theta-\frac{1}{\bar{t}}\right)^2\right]d\theta$$
$$=\frac{S}{2a}\left\{\Phi\left[\sqrt{n\lambda\bar{t}}\left(\mu+a-\frac{1}{\bar{t}}\right)\right]-\Phi\left[\sqrt{n\lambda\bar{t}}\left(\mu-a-\frac{1}{\bar{t}}\right)\right]\right\} \qquad (17)$$

where $S=\left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}}\prod_{i=1}^{n} t_i^{-\frac{3}{2}}e^{-\frac{\lambda\nu}{2}}\sqrt{\frac{2\pi}{n\lambda\bar{t}}}$. On differentiating Eqn (17) with respect to a, we have

$$\frac{d}{da}m(\underset{\sim}{t}\,|\,a)=-\frac{S}{2a^2}\left\{\Phi\left[\sqrt{n\lambda\bar{t}}\left(\mu+a-\frac{1}{\bar{t}}\right)\right]-\Phi\left[\sqrt{n\lambda\bar{t}}\left(\mu-a-\frac{1}{\bar{t}}\right)\right]\right\}$$
$$+\frac{S\sqrt{n\lambda\bar{t}}}{2a}\left\{\phi\left[\sqrt{n\lambda\bar{t}}\left(\mu+a-\frac{1}{\bar{t}}\right)\right]+\phi\left[\sqrt{n\lambda\bar{t}}\left(\mu-a-\frac{1}{\bar{t}}\right)\right]\right\} \qquad (18)$$

where $\phi(\cdot)$ denotes standard normal pdf.

Now substitute $z=\sqrt{n\lambda\bar{t}}\left|\frac{1}{\bar{t}}-\mu\right|$ and $a^*=a\sqrt{n\lambda\bar{t}}$ in Eqn (18) and equate to zero. The equation can be written as

$$a^*=z+\left\{-2\log\left[\sqrt{2\pi}\left(\frac{1}{a^*}\left[\Phi(a^*-z)-\Phi[-(a^*+z)]\right]-\phi[-(a^*+z)]\right)\right]\right\}^{\frac{1}{2}} \qquad (19)$$

Author solved Eqn (19) by standard fixed-point iteration, set $a^*=z$ on the right-hand side, which gives

$$\hat{a}=\begin{cases}0 & \text{if } z\le 1.65\\ \dfrac{a^*}{\sqrt{n\lambda\bar{t}}} & \text{if } z>1.65\end{cases}$$

the authors make $\hat{a}$ equal to zero when $\bar{t}$ is close to μ.

The usual Bayes point estimate $\hat{r}$, under quadratic loss function, is the posterior mean of $r(t_o;\theta,\lambda)$.

$$\hat{r}=\int_{\Theta} r(t_o;\theta,\lambda)\pi(\theta\,|\,\underset{\sim}{t},\lambda)\,d\theta$$

where posterior distribution $\pi(\theta \mid \underset{\sim}{t}, \lambda)$ of parameter θ wrt prior $\pi(\theta)$ is given by

$$\pi(\theta \mid \underset{\sim}{t}, \lambda) = \lambda(\underset{\sim}{t})\pi_o(\theta \mid \underset{\sim}{t}) + (1-\lambda(\underset{\sim}{t}))q(\theta \mid \underset{\sim}{t}) \tag{20}$$

The right hand side terms of Eqn (20) are evaluated as follows:

$$m(\underset{\sim}{t} \mid \pi_o) = S' \frac{G_1}{G} e^{-\beta} \;;$$

$$S' = \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \prod_{i=1}^{n} t_i^{-\frac{3}{2}} \sqrt{\frac{\tau}{\tau'}} \;;$$

$$G_1 = \Phi(-p'),\ p' = -\mu'\sqrt{\tau'},$$

$$m(\underset{\sim}{t} \mid q) = \frac{S}{2\hat{a}} \hat{\phi}_1 \;;$$

$$\hat{\phi}_1 = \Phi\left[\sqrt{n\lambda\bar{t}}\left(\mu + \hat{a} - \frac{1}{\bar{t}}\right)\right] - \Phi\left[\sqrt{n\lambda\bar{t}}\left(\mu - \hat{a} - \frac{1}{\bar{t}}\right)\right],$$

$$\pi_o(\theta \mid \underset{\sim}{t}) = \frac{1}{G_1}\sqrt{\frac{\tau'}{2\pi}} \exp\left[-\frac{\tau'}{2}(\theta - \mu')^2\right] \;;$$

$$\mu' = \frac{\tau\mu + n\lambda}{\tau'},\ \tau' = \tau + n\lambda\bar{t},$$

$$q(\theta \mid \underset{\sim}{t}) = \frac{1}{\hat{\phi}_1}\sqrt{\frac{n\lambda\bar{t}}{2\pi}} \exp\left[-\frac{n\lambda\bar{t}}{2}\left(\theta - \frac{1}{\bar{t}}\right)^2\right],$$

$$\lambda(\underset{\sim}{t}) = \left[1 + \frac{\varepsilon}{(1-\varepsilon)} \frac{G}{G_1}\left(\frac{n\lambda\tau\bar{t}}{2\pi\tau'}\right)^{\frac{-1}{2}} \frac{\hat{\phi}_1 e^{\beta'}}{2\hat{a}}\right]^{-1},$$

$$\beta = \beta' + \frac{\lambda v}{2} \text{ and } \beta' = \frac{n\lambda\tau\bar{t}}{2\tau'}\left(\mu - \frac{1}{\bar{t}}\right)^2.$$

thus

$$\hat{r} = \int_0^\infty \int_{t_o}^\infty \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} t^{-\frac{3}{2}} \exp\left[-\frac{\lambda t}{2}\left(\theta - \frac{1}{t}\right)^2\right] \pi_o(\theta \mid \underset{\sim}{t})\,dt\,d\theta$$

$$+ \int_{\mu-\hat{a}}^{\mu+\hat{a}} \int_{t_o}^\infty \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} t^{-\frac{3}{2}} \exp\left[-\frac{\lambda t}{2}\left(\theta - \frac{1}{t}\right)^2\right] q(\theta \mid \underset{\sim}{t})\,dt\,d\theta$$

$$= \frac{\lambda(\underset{\sim}{t})}{G_1}\left(\frac{\lambda\tau'}{2\pi}\right)^{\frac{1}{2}} \int_{t_o}^\infty t^{-\frac{3}{2}} \exp\left[-\frac{\lambda\tau' t}{2(\lambda t + \tau')}\left(\frac{1}{t} - \mu'\right)^2\right] \phi(\underset{\sim}{t})\,dt +$$

$$\frac{1-\lambda(\underset{\sim}{t})}{\hat{\phi}_1}\left(\frac{n\lambda^2\bar{t}}{2\pi}\right)^{\frac{1}{2}} \int_{t_o}^\infty t^{-\frac{3}{2}} \exp\left[-\frac{n\bar{t}\ t}{2(t + n\bar{t})}\left(\frac{1}{t} - \frac{1}{\bar{t}}\right)^2\right] \phi_1(\underset{\sim}{t})\,dt \tag{21}$$

where

$$\phi(\underset{\sim}{t}) = \Phi\left(\mu_1\sqrt{\lambda t + \tau'}\right),$$

$$\phi_1(\underset{\sim}{t}) = \Phi\left[\sqrt{\lambda(t + n\bar{t})}\left(\mu + \hat{a} - \mu_2\right)\right] - \Phi\left[\sqrt{\lambda(t + n\bar{t})}\left(\mu - \hat{a} - \mu_2\right)\right],$$

$$\mu_1 = \frac{\lambda + \mu'\tau'}{\lambda t + \tau'} \text{ and } \mu_2 = \frac{n+1}{t + n\bar{t}}.$$

The above two incomplete integrals in Eqn (21) are evaluated through numerical integration.

3.1 Lower One-sided Bayes Probability Interval Estimate

The authors constructed $100(1-\alpha)$ per cent lower one-sided Bayes probability interval(LBPI) estimate r_* of $r(t_o)$ where α is chosen to be a small quantity. Since $r(t_o;\theta,\lambda)$ is a monotonically non-decreasing function of θ for any fixed λ, one has the LBPI estimate of $r(t_o)$ as

$$r_* = \Phi\left(\sqrt{\frac{\lambda}{t_o}}\left(1 - t_o\theta^*\right)\right) - e^{2\lambda\theta^*}\Phi\left(-\sqrt{\frac{\lambda}{t_o}}\left(1 + t_o\theta^*\right)\right) \tag{22}$$

where θ^* is the $100(1-\alpha)$ per cent LBPI estimate of θ and is evaluated as

$$P(\theta \geq \theta^*) = \alpha$$

$$\int_{\theta^*}^{\Theta} \pi(\theta \mid \underset{\sim}{t}, \lambda)\,d\theta = \alpha$$

$$\lambda(\underset{\sim}{t})\int_{\theta^*}^{\infty} \pi_o(\theta \mid \underset{\sim}{t})\,d\theta + (1-\lambda(\underset{\sim}{t}))\int_{\theta^*}^{\mu+\hat{a}} q(\theta \mid \underset{\sim}{t})\,d\theta = \alpha$$

$$\frac{\lambda(\underset{\sim}{t})}{G_1}\Phi\left(-\sqrt{\tau'}(\theta^* - \mu')\right) - \frac{(1-\lambda(\underset{\sim}{t}))}{\hat{\phi}_1}\Phi\left(\sqrt{n\lambda\bar{t}}\left(\theta^* - \frac{1}{\bar{t}}\right)\right)$$

$$= \alpha - \frac{(1-\lambda(\underset{\sim}{t}))}{\hat{\phi}_1}\Phi\left(\sqrt{n\lambda\bar{t}}\left(\mu + \hat{a} - \frac{1}{\bar{t}}\right)\right)$$

The authors evaluated θ^* using Matlab for a given α and substituted in r_* to obtain the required LBPI for varying ε.

4. ILLUSTRATION

To study sensitivity of the Bayes reliability measure to the ML-II ε-contaminated prior to lognormal distribution, two sets of data were considered. Data-set 1 was the failure times (in hour) of the air conditioning system of 30 different airplanes obtained from Linhardt and Zucchini[22]. The data on active repair time (hour) are

Data-set 1
23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95.

Data-set 2 was considered from Barlow, Toland, and Freeman[23]. It represents the failure times on pressure vessels that were tested at 4300 psi. The complete ordered failure times were reported to be

Data-set 2
2.2, 4.0, 4.0, 4.6, 6.1, 6.7, 7.9, 8.3, 8.5, 9.1, 10.2, 12.5, 13.3, 14.0, 14.6, 15.0, 18.7, 22.1, 45.9, 55.4, 61.2, 87.5, 98.2, 101.0, 111.4, 144.0, 158.7, 243.9, 254.1, 444.4, 590.4, 638.2, 755.2, 952.2, 1108.2, 1148.5, 1569.3, 1750.6, 1802.1.

The precision ψ assumed known; ML estimate as its true value was taken. The subjective estimates of the parameters of the prior distribution were made on the basis of the above experiment.

For the inverse Gaussian distribution two sets of data were considered again. Data-set 3 is a simulated random sample of size $n = 30$ from IG population using algorithm given in Devrorye[24].

Data-set 3
0.45, 0.46, 0.66, 0.7, 0.94, 1.03, 1.29, 1.84, 1.89, 1.89, 1.91, 1.93, 1.93, 2.05, 2.1, 2.19, 2.74, 2.75, 3.18, 3.89, 4.26, 4.52, 4.56, 4.57, 4.94, 5.63, 7.67, 7.7, 26.78, 29.35

Data-set 4 was considered from Nadas[25]. Certain electronic device having thin film metal conductors failed due to mass depletion at a centre location on the conductor. The life time of such a device is the time elapsed until a critical amount of mass is depleted from the critical location. A sample of devices was tested under high stress conditions until all of these failed. There were $n = 10$ of these that were found to have failed due to mass depletion at the critical location. The corresponding lifetimes were summarised by the sufficient statistics $\bar{t} = 1.352$ and $\bar{t}_r = \frac{1}{n}\left(\sum_{i=1}^{n} t_i^{-1}\right) = 0.948$.

The prior parameter μ was taken to be approximately equal to the reciprocal of median of the IG(θ, λ) and precision τ equal to the reciprocal of the ML estimate of the variance. The value of known shape parameter λ was taken to be the ML estimate of $\hat{\lambda} = \left(\frac{n}{n-1}\left(\bar{t}_r - \frac{1}{\bar{t}}\right)\right)^{-1}$.

The Kolmogorov-Smirnov test statistic for the above three data-sets and the graphs of empirical and the theoretical curves are given in *Appendix 1*. The results show that lognormal fits well for data-sets 1 and 2 and inverse Gaussian is a good fit for data-set 3.

Bayesian results for Lognormal distribution in case of data-set 1 (n=30, ψ=0.5746, μ=4, t_0=10 h) are shown in Table 1 to Table 3, and in case of data-set 2 (n=39, ψ=0.2430, μ=5, t_0=100 h) are shown in Table 4 to Table 6. Bayesian results for inverse Gaussian distribution in case of Data-set 3 (n=30, μ=2.1450, λ=2.6339, t_0=5) are shown in Table 7 and 8, and in case of data-set 4 (n=10, μ=0.5, λ=4.8077, t_0=0.5) are shown in Table 9 and 10.

Tables 1-6 suggest that the Bayes reliability, LBPI and reliable life for lognormal distribution are not sensitive to contamination in the ML-II priors. Bayes reliability measures increase very little with the contamination increase in the priors at the various precision levels (τ). This variation is insignificant for both the data-sets 1 and 2 for varying precision, τ, and contamination, ε.

The Bayes reliability measures are insensitive to contaminations in the ML-II prior for inverse Gaussian distribution. Tables 7-10 suggest insignificant variation in Bayes reliability and LBPI for both the data-sets 3 and 4 for varying precision, τ, and contamination, ε, in the ML-II prior.

5. CONCLUSIONS

The numerical illustrations suggest that reasonable amount of misspecification in the prior distribution belonging to the class of ML-II ε-contaminated does not affect the

Table 1. Comparative values of Bayes reliability estimate for varying τ, ε

ε \ τ	0	0.05	0.2	0.5	0.9
0.01	0.784468	0.790316	0.795805	0.798426	0.799452
0.5	0.788418	0.789431	0.792010	0.795732	0.798973
0.9	0.791458	0.792068	0.793720	0.796390	0.799044

Table 2. Comparative values of Bayes LBPI (α = 0.05) estimate for varying τ, ε

ε \ τ	0	0.05	0.2	0.5	0.9
0.01	0.691529	0.717544	0.731520	0.736646	0.738472
0.5	0.697696	0.704984	0.717890	0.729984	0.737472
0.9	0.702446	0.708018	0.718791	0.729904	0.737421

Table 3. Comparative values of Bayes reliable life estimate for varying τ, ε (R=0.8.)

ε \ τ	0	0.05	0.2	0.5	0.9
0.01	9.749040	9.961537	10.161031	10.256270	10.293567
0.5	9.915409	9.950159	10.038584	10.166237	10.277391
0.9	10.046511	10.065425	10.116626	10.199404	10.281663

Table 4. Comparative values of Bayes reliability estimate for varying τ, ε

ε \ τ	0	0.05	0.2	0.5	0.9
0.01	0.398877	0.403509	0.409064	0.412279	0.413661
0.5	0.407347	0.407824	0.409125	0.411255	0.413407
0.9	0.413704	0.413713	0.413741	0.413790	0.413843

Table 5. Comparative values of Bayes LBPI (α=0.05) estimate for varying τ, ε

ε \ τ	0	0.05	0.2	0.5	0.9
0.01	0.214924	0.218005	0.224116	0.230643	0.235195
0.5	0.310689	0.315118	0.324176	0.334029	0.340984
0.9	0.318360	0.321578	0.328239	0.335787	0.341248

Table 6. Comparative values of Bayes reliable life estimate for varying τ, ε (R=0.8)

ε \ τ	0	0.05	0.2	0.5	0.9
0.01	11.292092	11.527624	11.810116	11.973591	12.043886
0.5	11.784220	11.803980	11.857837	11.946058	12.035160
0.9	12.165426	12.158146	12.137635	12.101767	12.062352

Table 7. Comparative values of Bayes reliability estimate for varying τ, ε

ε \ τ	0	0.05	0.2	0.5	0.9
0.01	0.269713	0.264824	0.261595	0.260360	0.259918
0.0284	0.269700	0.266013	0.262418	0.260654	0.259955
0.5	0.269356	0.267964	0.265091	0.262088	0.260173

Table 8. Comparative values of Bayes LBPI estimate for varying τ, ε (α = 0.05)

ε \ τ	0	0.05	0.2	0.5	0.9
0.01	0.182376	0.181409	0.180798	0.180571	0.180491
0.0284	0.182367	0.181636	0.180951	0.180624	0.180498
0.5	0.182135	0.181882	0.181373	0.180854	0.180531

Table 9. Comparative values of Bayes reliability estimate for varying τ, ε

ε \ τ	0	0.05	0.2	0.5	0.9
0.01	0.958766	0.961729	0.964049	0.965030	0.965395
0.05	0.958787	0.960600	0.963047	0.964623	0.965341
0.5	0.959022	0.959815	0.961597	0.963698	0.965188

Table 10. Comparative values of Bayes LBPI estimate for varying τ, ε (α = 0.05)

ε \ τ	0	0.05	0.2	0.5	0.9
0.01	0.927384	0.942010	0.947220	0.948853	0.949405
0.05	0.927389	0.938207	0.945238	0.948201	0.949323
0.5	0.927920	0.934339	0.939898	0.941203	0.940146

Bayesian reliability measures for lognormal and inverse Gaussian distributions. The mathematical results obtained in Sections 2 and 3 play down the effect of subjective choice of prior for the unknown parameters of both the distributions considered.

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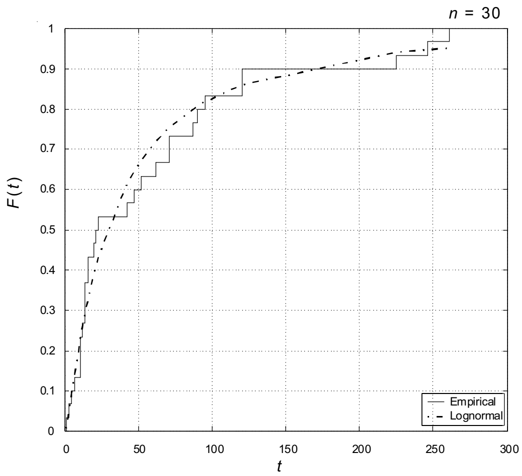
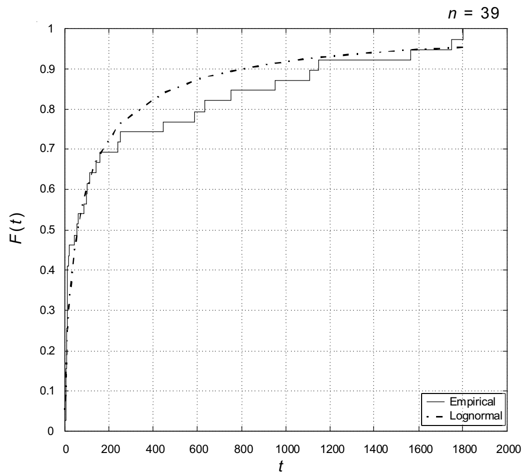
Appendix 1

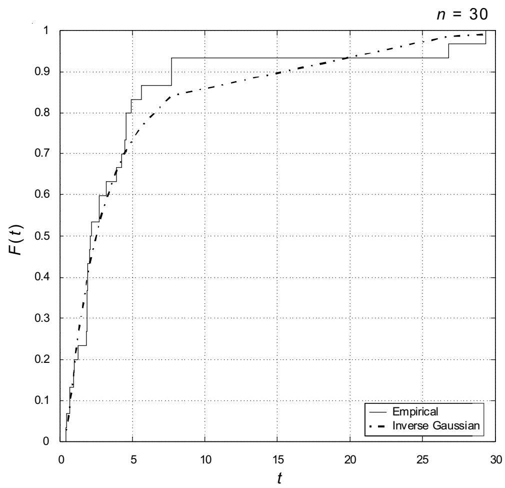
Empirical and theoretical curves for data-sets 1 and 2 for lognormal





	Kolmogorov – Smirnov Test and p sig. values		Decision at 5 per cent
	k-s	**p**	**0.05**
n=30	0.1047	0.8794	Data fits LN
n=39	0.1605	0.2450	Data fits LN

Empirical and theoretical curves for data-set 3 for Inverse Gaussian



	Kolmogorov – Smirnov Test and p sig. values		Decision at 5 per cent
	k-s	**p**	**0.05**
n=30	0.1535	0.4472	Data fits IG