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Optimum Non-Slender Geometries of Revolution for Minimum Drag in Free-Molecular Flow With Given Isoperimetric Constraints

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ABSTRACT

The problem of determining the non-slender bodies of revolution having minimum drag in free-molecular flow region for given integral constraints has been solved with the calculus of variations. Optimum bodies for known values of surface area and volume are blunt nosed and value of drag coefficients C_D decreases with the shape parameter Ⓗ $\left(= \frac{18\,\pi V^2}{S^3} \right)$. For given value of shape parameter, value of y_f decreases as K increases.

1. INTRODUCTION

Several investigations[1-7] have been made for the problem of determining the slender body of revolution having minimum drag in free-molecular flow with geometrical constraints, i.e. the length and thickness of the body are specified. Tan[8] has studied the problem for the non-slender body of revolution for given length and given diameter. Tawakley and Jain[9] determined optimum non-slender minimum drag bodies of revolution under the condition that one of the two geometrical quantities known must be an integeral constraints, i.e., either the surface area or the volume of the body is given in advance. The surface area and the volume have never been considered simultaneously as isoperimetric constraints. Here this problem of finding the non-slender bodies of revolution having minimum drag in free-molecular flow has been studied under given isoperimetric constraints, i.e., both surface area and volume of the body are together prescribed in advance. Results have also been compared with Large[10], who studied the same problem for hypersonic flow.

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2. FORMULATION OF THE EXACT PROBLEM

The drag function, assuming that the body has a flat nose of radius y_i and the drag due to the base is neglected, is given by

$$\frac{D}{2\pi q} = Ky_i^2 + y_f^2 + 2K \int_{x_i}^{x_f} \frac{yy'^2}{(1+y'^2)^{1/2}} dx \tag{1}$$

were q denotes the free stream dynamic pressure, x the axial distance, y(x) the radius of the body, y' the derivative dy/dx and K a constant dependent upon the temperature ratio and the speed ratio. The subscripts *i* and *f* refer to initial and final points respectively. The two integral constraints are given as the surface area of the body

$$\frac{S}{2\pi} = \int_{x_i}^{x_f} y(1+y'^2)^{1/2} dx + \frac{y_i^2}{2} \tag{2}$$

and the volume of the body

$$\frac{V}{\pi} = \int_{x_i}^{x_f} y'^2 dx \tag{3}$$

3. SOLUTION OF THE PROBLEM

Equation (1) can be written as

$$\frac{D}{2\pi q} = \int_{s_i}^{s_f} 2K\, yy'^2\, ds + y_f^2 + Ky_i^2 \tag{4}$$

where $y' = dy/ds$, where *ds* is the element along the arc.

Also

$$\frac{S}{2\pi} = \int_{s_i}^{s_f} y ds + \frac{y_i^2}{2} \tag{5}$$

$$\frac{V}{\pi} = \int_{s_i}^{s_f} y^2 (1-y'^2)^{1/2} ds \tag{6}$$

The fundamental function F in this case is

$$F \equiv 2Ky\dot{y}^2 + \lambda_1 y + \lambda_2 y^2 (1-\dot{y}^2)^{1/2} + G$$

Where

$$G \equiv K y_i^2 + y_f^2 + \lambda_1 \frac{y_i^2}{2}$$

λ_1, λ_2 are Lagrange multipliers. Since the fundamental function F does not contain the independent variable x explicitly, the Euler equation of the problem must admit the following first integral $F - \dot{y} F_{\dot{y}} = C$

$$\text{or} \quad 2Ky\dot{y}^2 - \lambda_1 y - \frac{\lambda_2 y^2}{(1-\dot{y}^2)^{1/2}} = C \tag{8}$$

where C is an integration constant

Now the transversality condition $[C\delta s + F_{\dot{y}}\delta y]_i^f + \delta G = 0$

becomes

$$\left[C\delta S + \frac{\lambda_2 y^2 \dot{y}}{(1-\dot{y}^2)^{1/2}} \delta_y \right]_i^f + \delta G = 0 \tag{9}$$

Since $s_i \neq 0$, we must have

$$C = 0$$

Also from Eqn. (9)

$$y_f \left(4K \dot{y}_f - \lambda_2 \frac{y_f \dot{y}_f}{(1-\dot{y}_f^2)^{1/2}} + 2\right) = 0$$

Since $y_f \neq 0$ (Physically impossible)

$$4Ky_f - \lambda_2 \frac{y_f y_f}{(1-\dot{y}_f^2)^{1/2}} + 2 = 0$$

and

$$y_i \left(4K \dot{y}_i - \lambda_2 \frac{\bar{y}_i \dot{y}_i}{(1-\dot{y}_i^2)^{1/2}} - 2K - \lambda_1\right) = 0$$

From Eqn. (12), if

$$y_i = 0 \tag{13}$$

Then, optimum curve passes through the origin.
Now from Eqns. (8) and (10), we get

$$y(2K\dot{y}^2 - \lambda_1 - \lambda_2 \frac{y}{(1-\dot{y}^2)^{1/2}}) = 0$$

Since y = 0 cannot be a solution, we must have

$$2K\dot{y}^2 - \lambda_1 - \lambda_2 \frac{y}{(1-\dot{y}^2)^{1/2}} = 0$$

or

$$y = \frac{(1-\dot{y}^2)^{1/2}}{\lambda_2} (2K\dot{y}^2 - \lambda_1) \tag{14}$$

and since

$$x = \int (1-\dot{y}^2)^{1/2} \frac{1}{y} \frac{dy}{d\dot{y}} \, d\dot{y}$$

Therefore,

$$x = \frac{1}{\lambda_2} [-2K\dot{y}^3 + (\lambda_1 + 4K)\dot{y} - (2K + \lambda_1)]_{\dot{y}_i}^{\dot{y}_f} \tag{14a}$$

If normalised co-ordinates, X = x/l, Y = 2 y/d are introduced, then

$$X = \frac{[-2K\dot{y}^3 + (\lambda_1 + 4K)\dot{y} - (2K + \lambda_1)]_{\dot{y}_i}^{\dot{y}}}{[-2K\dot{y}^3 + (\lambda_1 + 4K)\dot{y} - (2K + \lambda_1)]_{\dot{y}_i}^{\dot{y}_f}} \tag{15}$$

and

$$Y = \frac{(1-\dot{y}^2)^{1/2} (2K\dot{y}^2 - \lambda_1)}{(1-\dot{y}_f^2)^{1/2} (2K\dot{y}_f^2 - \lambda_1)} \tag{15a}$$

Which represent the parametric equation of the optimum curve
From Eqns. (11) and (14)

$$\lambda_1 = \frac{-2\{1 + y_i K (2 - \dot{y}_f^2)\}}{\dot{y}_f} \tag{16}$$

Equations (5) and (15a) give

$$\frac{S}{} = \frac{1}{\lambda_2^2}\left[-12/5\,K^2\,(\dot y_f^5 - \dot y_i^5) + 8/3K\,(\lambda_1 + K)\,(\dot y_f^3 - \dot y_i^3) - \lambda_1\,(\lambda_1 + 4K)\,(\dot y_f - \dot y_i)\right]$$

and from Eqns. (6) and (15a), we get

$$\frac{V}{\pi} = \frac{1}{\lambda_2^3}\left[\lambda_1^2\,(\lambda_1 + 4K)\,(\dot y_f - \dot y_i) - \frac{\lambda_1}{3}\,(\lambda_1^2 + 14\lambda_1 K + 16K^2)\,(\dot y_f^3 - \dot y_i^3) + 2K/5 + (\lambda_1 + 4K)\,(5\lambda_1 + 2K)\,(\dot y_f^5 - \dot y_i^5) - 4K^2/7\,(7\lambda_1 + 10K)\,(\dot y_f^7 - \dot y_i^7) + 8/3\,K^3\,(\dot y_f^9 - \dot y_i^9)\right] \tag{18}$$

These two relations lead to

$$\frac{18\,\pi\,V^2}{S^3} = 2.25\,[\lambda_1^2\,(\lambda_1 + 4K)\,(\dot y_f - \dot y_i) - \frac{\lambda_1}{3}(\lambda_1^2 + 14\lambda_1 K_1 + 16K^2)\,(\dot y_f^3 - \dot y_i^3) + 2K/5\,(\lambda_1 + 4K)\,(5\lambda_1 + 2K)\,(\dot y_f^5 - \dot y_i^5) - 4K^2/7\,(7\lambda_1 + 10K)\,(\dot y_f^7 - \dot y_i^7) + 8/3\,K^3\,(\dot y_f^9 - \dot y_i^9)]\,/\,[-12/5K^2\,(\dot y_f^5 - {}_i^5) + 8/3K\,(\lambda_1 + K)\,(\dot y_f^3 - \dot y_i^3) - \lambda_1\,(\lambda_1 + 4K)\,(\dot y_f - \dot y_i)]^3 \tag{19}$$

where $\frac{18\pi V^2}{S^3}$ (=Ⓗ) is defined as the shape parameter.

Now since $y_i = 0$, we have from Eqn. (14) that

$$\dot y_i = 1 \tag{20}$$

that is, initial slope at the nose is always a right angle. Now drag co-efficient

$$C_D = \frac{4D}{\pi q d^2}$$

or

$$C_D = 2\text{–}4K\,\frac{[12/7K^2\,(\dot y_f^7 - \dot y_i^7) - 8K/5(\lambda_1 + K)\,(\dot y_f^5 - \dot y_i^5) + \lambda_1/3\,(\lambda_1 + 4K)\,(\dot y_f^3 - \dot y_i^3)]}{(1 - \dot y_f^2)\,(2K\dot y_f^2 - \lambda_1)^2} \tag{21}$$

Where $\dot y_i$ =

Values of $\dot y_f$ for different values of shape parameter Ⓗ and K = 0.15 and 0.25 are obtained from Eqns. (16) and (19) and shown in Fig. 1. Minimum drag shapes Fig. 2 and Fig. 3 are drawn by using Eqns. (15) and (16) for various values of shape parameter and for known values of K. Relationship between C_D and shape paramter Ⓗ for K = 0.15, 0.2 and 0.25 is shown in Fig. 4.

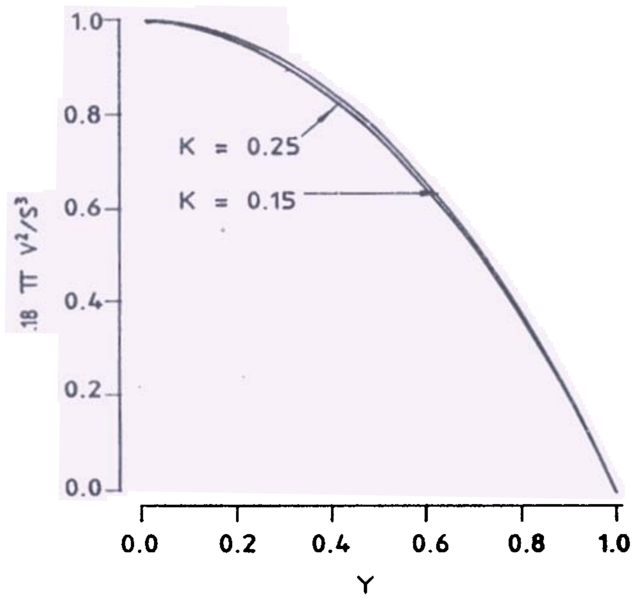

Figure 1. Values of y_f for different values of shape parameter Ⓗ and K = 0.15 and 0:.25.

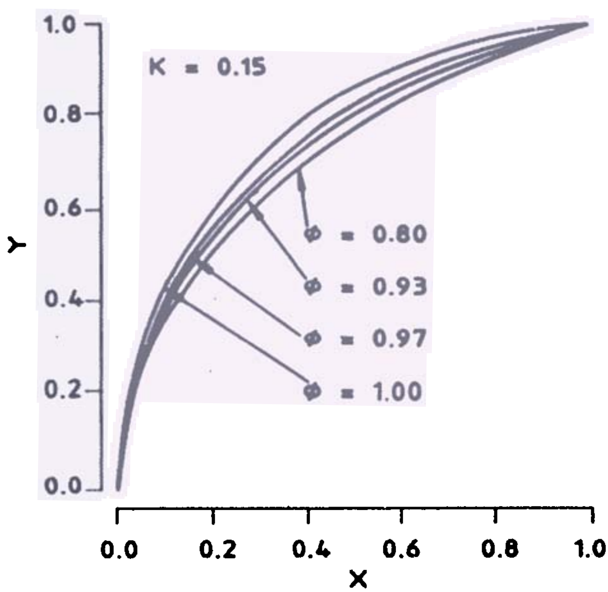

Figure 2. Minimum drag shapes are drawn for various values of shape parameter and known values of K = 0.15.

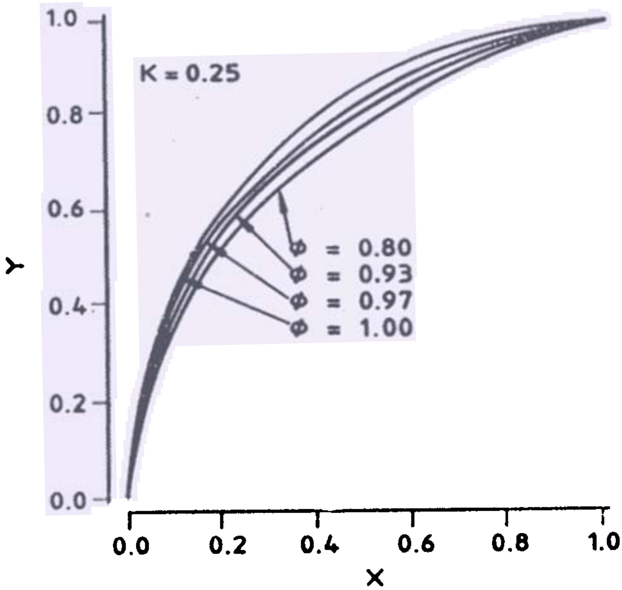

Figure 3. Minimum drag shapes are drawn for various values of shape parameter and known value of K = 0.25.

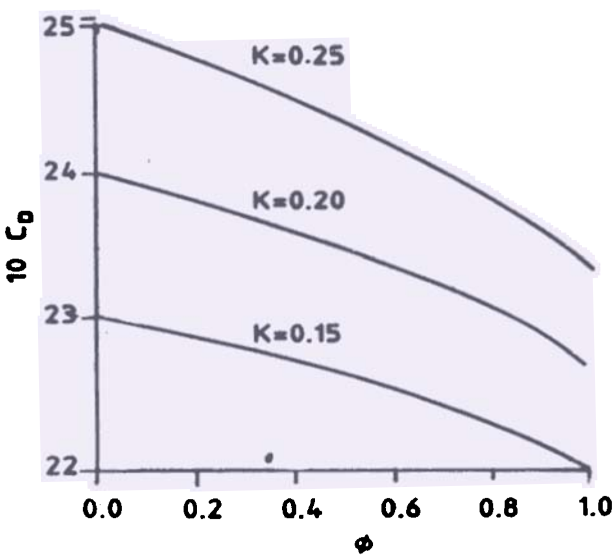

Figure 4. Relationship between C_D and shape parameter Ⓗ for K = 0.15, 0.2 and 0.25.

4. CONCLUSION

The problem of determining the optimum non-slender bodies of revolution in free molecular flow has been considered for specified isoperimetric constraints, i.e., for given surface area and volume of the body while diameter and length are free. It is shown that, if normalised co-ordinates are employed (that is, if the abscissa and the ordinate of the contour are measured in terms of the length of the body and radius of the base section respectively), optimum shapes for given shape parameter Ⓗ ($= 18\pi V^2/S^3$) are blunt nosed. As shape parameter Ⓗ increases, value of the drag co-efficient C_D decreases while in case of hypersonic flow it increases Large[10]. For fixed values of Ⓗ, value of y_f decreases and C_D increases as K increases. Optimum shape becomes hemisphere for all values of K, as in case of hypersonic flow for Ⓗ = 1.

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