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On the Kinematic Properties of Hydromagnetic Fluid Flows

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ABSTRACT

Herein assigning unidirection to magnetic field and using geometrical configuration of spatial curves of congruences formed by the streamlines, principal normals and their binormals, various kinetic and kinematic properties of hydromagnetic flows are established. Kinematic aspects of Bernoulli surface are also examined.

NOMENCLATURE OF SYMBOLS

μ	= Mean curvature
K	= Principal curvature along $\vec{s}$ lines
K'	= Principal curvature along $\vec{n}$ lines
K''	= Principal curvature along $\vec{b}$ lines
τ	= Torsion along $\vec{s}$ lines
σ'	= Torsion along $\vec{n}$ lines
σ''	= Torsion along $\vec{b}$ lines
$\vec{H}$	= Magnetic vector field
P	= Magnetic pressure
μ_e	= Magnetic permeability
B	= Bernoulli's function

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ρ	= Density
p	= Fluid pressure
q	= Velocity of fluid
M	= Mach number
E	= Sum of kinetic and magnetic energies
$\frac{d}{ds}, \frac{d}{dn}, \frac{d}{db}$	= Intrinsic differential operators along magnetic lines with $\vec{s}$, $\vec{n}$, $\vec{b}$ respectively.

1. INTRINSIC PROPERTIES OF FLOWS

Following S.I. Pai[1] and G. Purushotham[2], we write the basic equations governing hydromagnetic fluid flows in intrinsic form and study the various kinematic and kinetic properties of flows.

The equation of continuity assumes the form

$$K' + K'' = -\mu = \frac{d}{dS} \log (\rho q) \tag{1}$$

where μ is the mean curvature of stream surface and is equal to the sum of the curvature of normal congruences (K' and K'') of the streamline $(\vec{n} - \vec{b})$. If the stream surfaces are minimal[3] the flux per unit volume of mass is conserved along an individual streamline which is independent of the magnetic field.

For unidirection magnetic field $\vec{H} = \vec{h} H$, (where $\vec{h}$ is constant unit vector) conservation of H assumes

$$\frac{dH}{dh} = 0 \tag{2}$$

The Eqn. (2) can also be written as

$$\cos \alpha \frac{dH}{ds} + \cos \beta \frac{dH}{dn} + \cos \gamma \frac{dH}{db} = 0 \tag{3}$$

Where $\frac{d}{dh}$, $(\frac{d}{ds}, \frac{d}{dn}, \frac{d}{db})$ are the intrinsic differential operators along the magnetic lines and along the stream line triad $(\vec{s}, \vec{n}, \vec{b})$, the inclinations of the magnetic lines with this triad are a, β, γ respectively.

Defining the magnetic pressure as $P = p + \frac{1}{2} \mu_e H^2$ where p the fluid pressure and μ_e magnetic permeability. We transform the momentum equations as :

$$q \frac{dq}{ds} = -\frac{1}{\rho} \frac{dP}{dS} \tag{4}$$

$$Kq^2 = -\frac{1}{\rho} \frac{dP}{dn} \tag{5}$$

$$\frac{dp}{db} = 0 \tag{6}$$

These express the momentum relations along streamline triad, which are akin to the non-magnetic fluid flows, c.f. Prem Kumar[4]. From Eqn. (4), we see that the isovels and magnetic isobars touch the streamlines simultaneously as in the case of non-magnetic fluid flows. Streamlines are straight if magnetic isobars touch the principal normals of the streamlines, also the magnetic isobars envelop the binormal of the streamlines.

The Maxwell's field equations assume the following form :

$$\text{curl}\,\frac{d}{dS}(Hq) - H\nabla' q - \cos\alpha\,(K' + K'')\left(Hq - \lambda\frac{dq}{dS}\right)$$
$$= \lambda\left\{\cos\alpha\nabla_1^2 H + (K'\cos\beta + \sigma''\cos\alpha)\frac{dH}{dn} - \nabla'\left(\frac{dH}{dS}\right)\right.$$
$$\left. + (\sigma'\cos\beta - K''\cos\alpha)\frac{dH}{db}\right\} \tag{7}$$

$$\cos\beta\frac{d}{dS}(Hq) - (K\cos\alpha + K''\cos\beta + \sigma''\cos\gamma)\left(Hq - \lambda\frac{dH}{dS}\right)$$
$$= \lambda\left\{\cos\beta\nabla_1^2 H + \tau\cos\alpha\frac{dH}{db} - K\cos\beta\frac{dH}{dn} - \nabla'\left(\frac{dH}{dn}\right)\right\} \tag{8}$$

$$\cos\gamma\frac{d}{dS}(Hq) - (\cos\beta\sigma' + K'\cos\gamma)\left(Hq - \lambda\frac{dH}{dS}\right)$$
$$= \lambda\left\{\cos\gamma\nabla_1^2 H - (\cos\alpha\tau + K\cos\gamma)\frac{dH}{dn} - \nabla'\left(\frac{dH}{db}\right)\right\} \tag{9}$$

Where

$$\nabla_1^2 = \frac{d^2}{dS^2} + \frac{d^2}{dn^2} + \frac{d^2}{db^2}$$

$$\nabla' = \cos\alpha\frac{d}{dS} + \cos\beta\frac{d}{dn} + \cos\gamma\frac{d}{db}$$

These constitute the field equation with a finite electrical resistivity, in terms of curvatures of torsions of the curves of congruences. The same reduce to more elegant relations than those obtained by Suryanarayan[5] when the medium is of infinite electrical conductivity.

2. KINEMATIC ASPECTS OF BERNOULLI FUNCTION

Introducing the Bernoulli function, the momentum equation can be written as :

$$\nabla B = -\frac{1}{\rho}\left\{\nabla p + \frac{1}{2}\nabla(\rho q^2)\right\} + \frac{q^2}{2}\nabla\log\rho \tag{10}$$

Where the normals to Bernoulli surface are given by

$$\nabla B = \vec{n}q\left(Kq - \frac{dq}{dn}\right) - \vec{b}q\frac{dq}{db} \qquad [11]$$

It is evident from this that the Bernoulli surface contains streamlines, also denoting $\frac{d}{dN}$ as the directional derivative along the normal to the Bernoulli surface, we have

$$\frac{dB}{dN} = q\left\{\left(\frac{dq}{db}\right)^2 + \left(Kq - \frac{dq}{dn}\right)^2\right\}^{1/2}$$

Hence a necessary and sufficient condition that the streamlines to be a family of parallel curves is that $q^2\left\{\left(\frac{dq}{db}\right)^2 + \left(Kq - \frac{dq}{dn}\right)^2\right\}$ is constant along an individual Bernoulli surfaces. The same has been observed for non-magnetic flows[5]. The vorticity vector in intrinsic form is[4]

$$\text{curl } \vec{q} = \vec{t}q(\sigma' - \sigma'') + \vec{n}\frac{dq}{db} + \vec{b}\left(Kq - \frac{dq}{dn}\right) \tag{12}$$

Also using solenoidal property of the vorticity and curl, we obtain either

$$\frac{d\vec{q}}{dS} = 0 \tag{13}$$

or

$$\sigma' - \sigma'' = 0 \tag{14}$$

From these, we infer that the magnitude of the velocity is either uniform along streamline or it is complex lamellar. Forming scalar product of Eqns. (12) and (11), we get

$$\nabla B \cdot \text{curl } \vec{q} = 0 \tag{15}$$

This shows that the Bernoulli surfaces also contain the vortex lines or in other words the vortex lines and streamlines intersect orthogonally along a Bernoulli surface. The same has been observed by Suryanarayan[6] for non-magnetic flows.

The Eqn. (10) can also be written as

$$\nabla B = -\frac{1}{\rho}\nabla(p + E) + \frac{q^2}{2\rho}\nabla\rho \tag{16}$$

Where $E = \frac{\mu_e}{2}H^2 + \frac{\rho}{2}q^2$, which is the sum of kinetic and magnetic energies.

Introducing Mach number in Eqn. (16), we obtain

$$2\frac{dE}{dS} = q^2\left(1 - \frac{2}{M^2}\right)\frac{d\rho}{dS} \tag{17}$$

We observe that the sum of kinetic and magnetic energies shall remain constant along an individual streamline either in supersonic region or density remains uniform in the same direction. Also from the relations (11) and (16), we observe for incompressible flow that the sum of kinetic and magnetic energies remains uniform along a streamline only when the hydrodynamic pressure is uniform along the same direction.

From Eqns. (1), (4) and (17), we obtain

$$\frac{dq}{dS} + \mu q = \frac{qH\mu_e}{\rho c^2}\frac{dH}{dS} + M^2\frac{dq}{dS} \tag{18}$$

Eliminating $\frac{dq}{ds}$ from Eqns. (4) and (18), we get

$$\frac{1}{\rho q}\frac{dp}{ds} = \frac{1}{M^2 - 1}\left(\frac{\mu_e H}{\rho q^2}\frac{dH}{dS} - \mu\right) \tag{19}$$

This equation generalizes Kanwal's[7] result for magnetogasdynamic flow.

From Eqns. (4) and (19), we observe that if the surfaces are minimal and the magnetic field is uniform along a streamline, then the pressure, velocity and density are also constant along the same direction. Making use of this result in Eqn. (5), we observe the streamlines to be either right circular helices or circles or parallel lines.

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