Def Sci J, Vol 38, No. 1, January 1988, pp 21-27

Oscillating Flow of a Viscous Liquid in a Porous Rectangular Duct

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ABSTRACT

The unsteady flow of viscous incompressible liquid in a long porous straight duct of rectangular cross-section, under the influence of periodic pressure gradient using the generalised momentum equation, has been studied. The finite cosine transforms have been employed to solve the problem. Expressions for velocity distribution, volume flow rate (flux) and drag in the duct have been derived. A few particular cases have been deduced. It is found that the classical Darcian effect is felt only in a core very near to the axis of the duct and the non-Darcian phenomenon is felt predominently, near the boundary of the duct. It is also found that the velocity, volume flow rate (flux) and drag increase with the increase in frequency of oscillations of liquid. Whereas the porosity of the medium reduces both the velocity and flux and increases the drag.

1. INTRODUCTION

Flow of a viscous liquid in a porous medium is of great and increasing importance in the study of percolation through soils in hydrology, petroleum industry and in agricultural engineering. Henry Darcy had observed while studying flow of water through sand filters that the flow rate of water is proportional to the difference in head of water across the filter and the cross-sectional area of bed. Subsequently, many experiments were conducted to study the flow of various fluids through different types of porous solids.

Pattabhiramacharyulu[1] has discussed the steady flow of viscous liquid in a porous circular tube using the generalised Darcy's law for flow through highly porous media :

Received 4 July 1985

$$0 = \nabla p - \frac{\mu}{K}\vec{V} + \mu\nabla^2\vec{V} \tag{1}$$

as proposed by Brinkman[2] for steady flow, where $\vec{v}$ and P represent velocity and pressure fields, μ is viscosity co-efficient of liquid and K is the permeability of the medium. Later Narasimhacharyulu and Pattabhiramacharyulu[3] extended this problem to elliptic tube. Tam[4] derived analytically the same equation to study the flow past spherical particles at low Reynold's number. Yamamoto[5,6] examined flow past porous bodies applying the generalised law. Johri[7] has studied the unsteady flow of viscous liquid in a porous circular tube under the influence of periodic pressure gradient using the generalised momentum equation.

In this paper, we have considered the flow of viscous incompressible liquid through a long porous straight rectangular duct under the influence of a periodic pressure gradient using the generalised momentum equation.

2. FORMATION OF THE PROBLEM AND SOLUTION

Consider the flow in a straight duct of rectangular cross-section in xy-plane. The z-axis is parallel to the length of the duct with impermeable boundary B; $x = -a$, $x = a$; $y = -b$, $y = b$. The velocity components in x, y, z directions are taken to be 0, 0, $w(x, y, t)$ respectively, where $w(x, y, t)$ is the axial velocity of the liquid.

The equation of continuity

$$\text{div } \vec{v} = 0 \tag{2}$$

is satisfied with the choice of the velocity. The equation for unsteady motion, following generalised Darcy's law[1], is given by

$$\frac{\partial W}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 W - \frac{\nu}{K}W \cdot (\nu = \mu/\rho) \tag{3}$$

Introducing the non-dimensional quantities,

$$\left.\begin{aligned}(\bar{W}, \bar{W}_1) &= (W, W_1)\frac{a}{\nu},\ \bar{p} = \frac{pa^2}{\rho\nu^2},\ \bar{t} = \frac{t\nu}{a^2}\\ \bar{\omega} &= \frac{a^2}{\nu}\omega,\ (\bar{x}, \bar{y}, \bar{z}) = \frac{1}{a}(x, y, z)\end{aligned}\right\} \tag{4}$$

Eqn. (3) is reduced to

$$\frac{\partial \bar{W}}{\partial \bar{t}} = -\frac{\partial \bar{p}}{\partial \bar{z}} + \nabla^2\bar{W} - \frac{a^2}{K}\bar{W}, \tag{5}$$

Where, $-\dfrac{\delta \bar{p}}{\delta \bar{z}}$ is the axial pressure gradient.

Initially, the liquid is at rest and we have used the fact that the momentum flux, the pressure gradient, the local velocity and the volume flow rate are all periodic in time with frequency $\bar{\omega}$, 'periodic' means that the local fluid motion is Sinusoidal function of time.

Assuming

$$\left.\begin{aligned} -\frac{\partial \bar{p}}{\partial \bar{z}} &= Re\ A\ e^{+i\bar{\omega}\bar{t}} \\ \bar{W} &= R_e\,\bar{W}_1(\bar{x}, \bar{y})\, e^{-i\bar{\omega}\bar{t}}\ Q = R_e Q_0\, e^{+i\bar{\omega}\bar{t}} \end{aligned}\right\} \quad (6)$$

Where, Q is flux and A is real, whereas $\overline{W}_1$ and Q_0 are complex. Substituting for $\overline{W}$ from Eqn. (6) in Eqn. (5), we get.

$$\nabla^2 \bar{W}_1 - \left(\frac{a^2}{K} + i\bar{\omega}\right)\bar{W}_1 = -A \quad (7)$$

For symmetric consideration, the flow in region $\bar{x} \geqslant 0$, $\bar{y} \geqslant 0$, is considered. Accordingly, the boundary conditions are :

$$\left.\begin{aligned} \bar{t} > 0 : \bar{W}(1, \bar{y}, \bar{t}) &= 0,\ 0 \leqslant \bar{y} \leqslant \frac{b}{a} \\ \frac{\partial \bar{W}}{\partial \bar{x}} &= 0 \quad \text{at} \quad \bar{x} = 0 \end{aligned}\right\} \quad (8)$$

and

$$\left.\begin{aligned} \bar{W}\left(\bar{x}, \frac{b}{a}, \bar{t}\right) &= 0, \quad 0 \leqslant \bar{x} \leqslant 1 \\ \frac{\partial \bar{W}}{\partial \bar{y}} &= 0 \quad \text{at } \bar{y} = 0 \end{aligned}\right\} \quad (9)$$

3. SOLUTION OF THE PROBLEM

To solve the problem, we choose the finite cosine transform defined by

$$\bar{\bar{W}}(m, \bar{y}, \bar{t}) = \int_0^1 \bar{W}(\bar{x}, \bar{y}, \bar{t}) \cos q_m \bar{x}\, d\bar{x} \quad (10)$$

and

$$\bar{\bar{W}}(\bar{x}, n, \bar{t}) = \int_0^{b/a} \bar{W}(\bar{x}, \bar{y}, \bar{t}) \cos q_n \bar{y}\, d\bar{y}, \quad (11)$$

Where

$$q_m = \frac{(2m+1)\pi}{2}, \quad q_n = \frac{(2n+1)\pi a}{2b} \quad (12)$$

Now, multiplying Eqn. (7) by $Cos\ q_m\bar{x}.Cos\ q_n\bar{y}$ and then integrating twice with respect to $\bar{x}$ and $\bar{y}$ in the units from $\bar{x} = 0$ to $\bar{x} = 1$, and $\bar{y} = 0$ to $\bar{y} = \frac{b}{a}$ and using the boundary conditions (8) and (9), we get

$$\bar{\bar{W}}_1(m, n) = \frac{A(-1)^{m+n}}{q_m q_n(\alpha_{m,n} + i\bar{\omega})}$$

Applying the inversion formula for the finite cosine transform defined by

$$\begin{aligned} \bar{W}_1(\bar{x}, \bar{y}) &= \frac{4a}{b}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \bar{W}_1(m, n) \cos q_m\bar{x} \cos q_n\bar{y} \\ &= \frac{4aA}{b}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{m+n}\cos q_m\bar{x}\cos q_n\bar{y}}{q_m q_n(\alpha_{m,n} + i\bar{\omega})} \end{aligned} \quad (13)$$

Where $\quad \alpha_{m,n} = q_m^2 + q_n^2 + \dfrac{a^2}{K}$

Therefore,

$$\bar{W}(\bar{x}, \bar{y}, \bar{t}) = R_e \cdot \bar{W}_1(\bar{x}, \bar{y})\, e^{+i\bar{\omega}\bar{t}}$$

$$= \frac{4aA}{b} \sum_0^\infty \sum_0^\infty \frac{(-1)^{m+n}(\alpha_{m,n} \cos \bar{\omega}\bar{t} + \bar{\omega} \sin \bar{\omega}\bar{t}) \times \cos q_m\bar{x}}{q_m q_n(\alpha_{m,n}^2 + \bar{\omega}^2)} \cos q_n\bar{y} \quad (14)$$

Now

$$Q_0 = \int_{-a}^{a}\int_{-b}^{b} W_1(x, y)\, dx\, dy = \nu a \int_{-1}^{1} \int_{-b/a}^{b/a} \bar{W}_1(\bar{x}, \bar{y})\, d\bar{x}\, d\bar{y}$$

$$= \frac{16\nu a^2 A}{b} \sum_0^\infty \sum_0^\infty \frac{1}{(q_m q_n)^2(\alpha_{m,n} + i\bar{\omega})}$$

Hence

$$Q = R_e \cdot Q_0\, e^{+i\bar{\omega}\bar{t}}$$

$$= \frac{16\nu a^2 A}{b} \sum_0^\infty \sum_0^\infty \frac{[\alpha_{m,n} \cos \bar{\omega}\bar{t} + \bar{\omega} \sin \bar{\omega}\bar{t}]}{(q_m q_n)^2(\alpha_{m,n}^2 + \bar{\omega}^2)} \quad (15)$$

The drag D per unit length of the duct is given by

$$D = \int_c T_{nz}\, ds,$$

Where

T_{nz} is the stress and $\vec{\eta}$ is the outward drawn normal and C is the contour of cross-section area S (= 4 ab). By Green's theorem, we have from Eqn. (3) that

$$D = \iint_S \left(\frac{\partial Txz}{\partial x} + \frac{\partial Tyz}{\partial y}\right) dx\, dy$$

$$= \mu \int_{-a}^{a} \int_{-b}^{b} \left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right) dx\, dy$$

$$= \frac{\mu\nu}{b} \int_{-1}^{1} \int_{-b/a}^{b/a} (\nabla^2 \bar{W})\, d\bar{x}\, d\bar{y}$$

$$= \frac{16\mu\nu A}{b} \sum_0^\infty \sum_0^\infty \frac{(\alpha_{m,n} \cos \bar{\omega}\bar{t} + \bar{\omega} \sin \bar{\omega}\bar{t})(-1)^{n+m}}{(q_m q_n)^3(\alpha_{m,n}^2 + \bar{\omega}^2)} \quad (16)$$

Where q_m, q_n and $a_{m,n}$ are given by Eqns. (12) and (13) respectively.

4. DEDUCTIONS

(I) When K is very small, i.e., the medium is highly porous.

In that case the velocity, flux and drag are given by

$$\bar{W}(\bar{x}, \bar{y}, \bar{t}) = \frac{4A}{a^3 b} \sum_0^\infty \sum_0^\infty \frac{(-1)^{m+n} K}{q_m q_n}[a^2 \cos \bar{\omega}\bar{t} + K\{\bar{\omega} \sin \bar{\omega}\bar{t}$$

$$-(q_m^2 + q_n^2) \cos \bar{\omega}\bar{t}\}] \times \cos q_m\bar{x} \cdot \cos q_n\bar{y} \quad (17)$$

$$Q = \frac{16\nu A}{a^2 b} \sum_0^\infty \sum_0^\infty \frac{K}{(q_m q_n)^2} [a^2 \cos \bar{\omega} \bar{t} + K\{\bar{\omega} \sin \bar{\omega} \bar{t} - (q_m^2 + q_n^2) \cos \bar{\omega} \bar{t}\}] \tag{18}$$

$$D = \frac{16\mu\nu A}{a^4 b} \sum_0^\infty \sum_0^\infty \frac{(-1)^{m+n} K}{(q_m q_n)^3} [a^2 \cos \bar{\omega} \bar{t} + K\{\bar{\omega} \sin \bar{\omega} \bar{t} - (q_m^2 + q_n^2) \cos \bar{\omega} \bar{t}\}] \tag{19}$$

respectively.

(II) When $K \to \infty$, i.e., when no resistance is offered by the medium

In that case $\bar{W}$, Q and D are reduced to

$$\bar{W}(\bar{x}, \bar{y}, \bar{t}) = \frac{4aA}{b} \sum_0^\infty \sum_0^\infty \frac{(-1)^{m+n}[(q_m^2 + q_n^2) \cos \bar{\omega} \bar{t} + \bar{\omega} \sin \bar{\omega} \bar{t}]}{(q_m q_n)[(q_m^2 + q_n^2)^2 + \bar{\omega}^2]} \times \cos q_m \bar{x} \cdot \cos q_n \bar{y} \tag{20}$$

$$Q = \frac{16\nu a^2 A}{b} \sum_0^\infty \sum_0^\infty \frac{[(q_m^2 + q_n^2) \cos \bar{\omega} \bar{t} + \bar{\omega} \sin \bar{\omega} \bar{t}]}{(q_m q_n)^2[(q_m^2 + q_n^2)^2 + \bar{\omega}^2]} \tag{21}$$

$$D = \frac{16\mu\nu A}{b} \sum_0^\infty \sum_0^\infty \frac{(-1)^{m+n}[(q_m^2 + q_n^2) \cos \bar{\omega} \bar{t} + \bar{\omega} \sin \bar{\omega} \bar{t}]}{(q_m q_n)^3[(q_m^2 + q_n^2)^2 + \bar{\omega}^2]} \tag{22}$$

(III) The case of steady flow ($\bar{\omega} = 0$,)

For K to be very small, $\bar{W}$, Q and D are given by

$$\bar{W} = \frac{4aAK}{ba^4} \sum_0^\infty \sum_0^\infty \frac{(-1)^{m+n}}{q_m q_n} [a^2 + K(q_m^2 + q_n^2)] \cos q_m \bar{x} \cos q_n \bar{y} \tag{23}$$

$$Q = \frac{16\nu AK}{a^2 b} \sum_0^\infty \sum_0^\infty \frac{[a^2 + K(q_m^2 + q_n^2)]}{(q_m q_n)^2} \tag{24}$$

$$D = \frac{16\nu\mu AK}{a^4 b} \sum_0^\infty \sum_0^\infty \frac{[a^2 + K(q_m^2 + q_n^2)](-1)^{m+n}}{(q_m q_n)^3} \tag{25}$$

For $K \to \infty$, we have

$$\bar{W} = \frac{4aA}{b} \sum_0^\infty \sum_0^\infty \frac{(-1)^{m+n} \cos q_m \bar{x} \cos q_n \bar{y}}{(q_m q_n)(q_m^2 + q_n^2)} \tag{26}$$

$$Q = \frac{16\nu a^2 A}{b} \sum_0^\infty \sum_0^\infty \frac{1}{(q_m q_n)^2 (q_m^2 + q_n^2)} \tag{27}$$

$$D = \frac{16\mu\nu A}{b} \sum_0^\infty \sum_0^\infty \frac{(-1)^{m+n}}{(q_m q_n)^3 (q_m^2 + q_n^2)} \tag{28}$$

(IV) When $\bar{x} = 0$, $\bar{y} = 0$, the velocity $\overline{W}(\bar{x}, \bar{y}, \bar{t})$ in the case, when K is very small is given by

$$\overline{W} = \frac{4AK}{ba^3} \sum_0^\infty \sum_0^\infty \frac{(-1)^{m+n}[a^2 \cos \bar{\omega}\bar{t} + K\{\bar{\omega} \sin \bar{\omega}\bar{t} - (q_m^2 + q_n^2) \cos \bar{\omega}\bar{t}\}]}{q_m q_n} \tag{29}$$

5. DISCUSSION AND CONCLUSION

Expressions (14), (15) and (16) have been derived the express velocity of flow, volume flow rate and drag on the walls of the ducts. Eqns. (17), (18) and (19) have been computed numerically verses $\bar{\omega}$ for different values of the permeability of the medium K (= 0.01, 0.05, 0.10) and have been shown in Figs. 1, 2 and 3. From these figures it is evident that the velocity, volume flow rate and drag increase with increasing frequency of oscillations of the liquid whereas the porosity of the medium reduces both the velocity and flux and increases the drag.

Eqn. (29) is same as may be obtained under the classical Darcy's law (Mushakat)[8].

$$\frac{\partial \overline{W}}{\partial \bar{t}} = -\frac{\partial \bar{p}}{\partial \bar{z}} - \frac{a^2}{K}\overline{W}$$

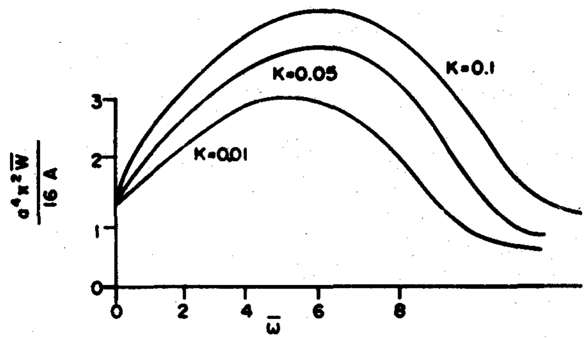

Figure 1. Velocity profiles ($\overline{W}$ vs. $\bar{\omega}$) for $\bar{t} = 0.1$, $\bar{x} = 2$ and $\bar{y} = \frac{2b}{a}$.

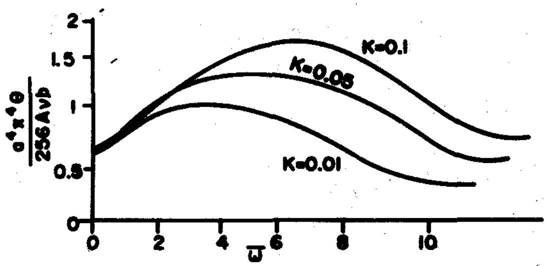

Figure 2. Variations of volume flow rate Q vs. $\bar{\omega}$ when $\bar{t} = 0.1$, $\bar{x} = 2$ and $\bar{y} = \frac{2b}{a}$.

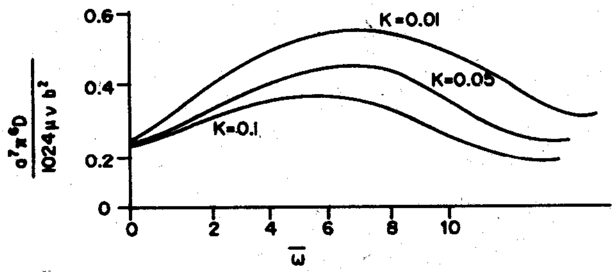

Figure 3. Drag (D) per unit length on the walls of the duct, when $\bar{t} = 0.1$, $\bar{x} = 2$ and $\bar{y} = \frac{2b}{a}$.

Hence the classical Darcian effect is felt only in a core very near to the axis of the duct and the non-Darcian phenomenon is felt predominently near the boundary of the duct.

Eqns. (20), (21) and (22) expresses velocity, flux and drag when no resistance is offered by the medium, i.e., when rectangular duct is not porous. Eqns. (23) to (28) expressing velocity flux and drag when the flow is steady.

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