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Unsteady Porous Channel Flow of a Conducting Fluid with Suspended Particles

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ABSTRACT

The flow of a viscous incompressible fluid embedded with a small spherical particle in the presence of a transverse magnetic field in a channel has been discussed. The cross-section of the channel is a porous regular hexagonal of side 4a and the walls are non-conducting. The analysis applied to the flows with pressure gradient which are arbitrary function of time. A few particular cases, flow for impulsive pressure gradient and for constant pressure gradient have been studied. The velocity of the fluid and particle decrease with increase in the intensity of the magnetic field.

NOMENCLATURE

H_0 = strength of the imposed magnetic field

k = $N_0 m/\rho$ density ratio of particles to fluid (per unit volume of flow field)

τ = K/ν particle relaxation time

N_0 = the number density of particles

m = the mass of a particle

K = the Stoke's resistance coefficient

M = $\mu_e^2 H_0^2/\rho$

t = time

u = gas velocity

v = particle velocity

μ_e = magnetic permeability of the gas

γ = kinematic viscosity of the gas

σ = electrical conductivity of the gas

ρ = density of the gas

p = pressure

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1. INTRODUCTION

In recent years, a number of studies of fluid embedded with particles have been appeared in literature. The study of fluid or gas having uniform distribution of solid particles is of interest in a wide range of technical problems. These include fluidization, environmental pollution, combustion and more recently blood flow.

Rao[1,2] has discussed the flow of a dusty gas in a circular cylinder under the influence of a pressure gradient which varies exponentially with time. Lu and Miller[3] have discussed the flow of the dusty gas under the influence of a constant pressure gradient in a channel with equitriangular cross-section. Singh and Pathak[4] have considered the flow of the dusty viscous fluid in a tube with sector of a circle as cross-section under the influence of exponen-pressure gradient. Recently, Gupta and Gupta have studied the flow of the dusty gas in a rectangular channel with arbitrary time varying pressure gradient. Gupta has discussed the unsteady channel flow of a conducting fluid with suspended particles.

Experimental law to study the flow through porous media was first of all given by Darcy. Brinkman[5] generalized the Darcy's law to study the flow through highly porous media. Recently, Ahmadi and Manvi[6] derived a general equation of motion for flow through porous media and applied the results obtained to solve some basic flow problems. Gulab Ram and Mishra[7] have studied the MHD flow of a viscous fluid through porous circular tube.

Here we are studying the flow of an incompressible viscous electrically conducting fluid, embedded with small inert particles in the presence of transverse magnetic field of a uniform intensity in a channel whose cross-section is an porous regular hexagonal duct with impermeable boundary and under time varying axial pressure gradient.

Non-circular ducts are frequently used in automobile radiators, nuclear power plants, aerospace vehicles etc., as they can be fitted within the available space between compactly placed components. The flow fluid embedded with particles in such ducts is of great importance. One finds that the behaviour of oil or fuel inducts due to presence of particles is considerably changed and many new phenomena can be observed.

2. FORMULATION OF THE PROBLEM

The appropriate equations of motion for the problem[8] (Fig. 1) are given by :

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \gamma\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{k}{\tau}(v - u) - \left(m + \frac{\gamma}{k}\right)u \qquad (1)$$

$$\frac{\partial V}{\partial t} = \frac{1}{\tau}(u - v) \qquad (2)$$

The suspended particles are very small and the Reynolds number of the relative motion of particles and gas is smaller than unity[8]. The electromagnetic effect and the

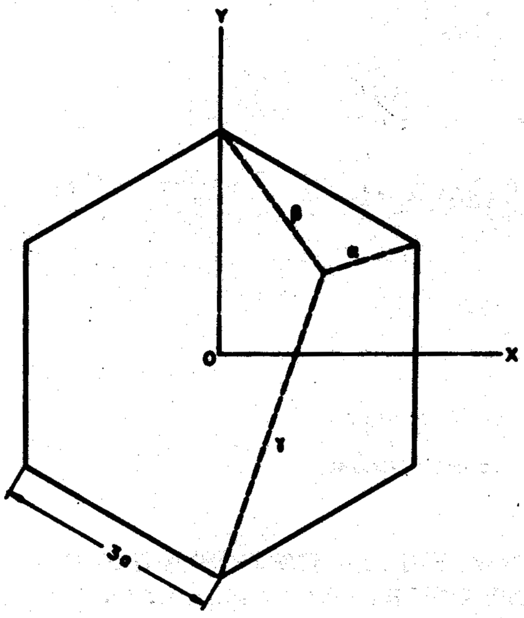

Figure 1. Cross-section of the tube

porous parameter entering the problem in last term of Eqn. (1) is an approximation under the suspension of small electric conductivity.

Now let,

$$-\frac{1}{\rho}\frac{\delta P}{\delta Z} = \text{any function of time} = f(t)$$

Eliminating v from Eqns. (1) and (2), we have

$$\frac{\partial^2 u}{\partial t^2} + \left\{\frac{k}{\tau} + \frac{1}{\tau} + \frac{m\gamma}{\tau}\left(\frac{1}{k'} + M^2\right)\right\}\frac{\partial u}{\partial t}$$

$$= \left(\frac{1}{\tau} + \frac{\partial}{\partial t}\right) f(t) + \gamma\left(\frac{1}{\tau} + \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (4)$$

Also let $f(t) = P + F(t)$ (5)

where P is the constant and $F(t)$ is the function of time.

Let $u = u_1(x, y) + u_2(x, y, t)$ (6)

where u_1 is the steady component and u_2 is the unsteady component of velocity of the fluid.

Substituting Eqns. (4) and (5) in Eqn. (6), and separating the steady and unsteady parts, we get

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} + \frac{P}{\gamma} = 0 \tag{7}$$

$$\frac{\partial^2 u_2}{\partial t^2} + \left\{\frac{k}{\tau} + \frac{1}{\tau} + \frac{m\gamma}{\tau}\left(\frac{1}{k'} + M^2\right)\right\} \frac{\partial u_2}{\partial t}$$

$$= \left(\frac{1}{\tau} + \frac{\partial}{\partial t}\right) F(t) + \gamma\left(\frac{1}{\tau} + \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2}\right) \tag{8}$$

Boundary conditions are

(i) $u_2 = 0$ at $t \leqslant 0$ (9)

(ii) $u_2 = 0$ at $t > 0$ on the boundary (10)

$u_1 = w$ at $t > 0$ on the boundary.

3. TRANSFORMATION OF THE GOVERNING EQUATIONS AND BOUNDARY CONDITIONS IN TRILINEAR COORDINATES

Let us consider a reference equilateral triangle of side *3a*, so that the hexagonal cross-section has three alternative sides as the sides of the reference triangle and remaining three alternative sides will be given by constant perpendicular distances from the reference triangles, sides. Hence the hexagon will have its sides of length *a* expressing the equation of motion and boundary conditions in terms of system of trilinear coordinates, we have

$$\nabla^2_{\alpha,\beta,\gamma} u_1 + \frac{P}{\gamma} = 0 \tag{11}$$

$$\frac{\partial^2 u_2}{\partial t^2} + \left(\frac{k}{\tau} + \frac{1}{\tau} + \frac{m\gamma}{\tau}\left(\frac{1}{k'} + M^2\right)\right\} \frac{\partial u_2}{\partial t}$$

$$= \left(\frac{1}{\tau} + \frac{\partial}{\partial t}\right) F(t) + \gamma\left(\frac{1}{\tau} + \frac{\partial}{\partial t}\right) \nabla^2_{\alpha,\beta,\gamma} u_2 \tag{12}$$

with boundary conditions

$$u_1(0, \beta, \gamma) = w = u_1\left(\frac{2K^n}{3}, \beta, \gamma\right)$$

$$u_1(\alpha, 0, \gamma) = w = u_1\left(\alpha, \frac{2K^n}{3}, \gamma\right) \tag{13}$$

$$u_1(\alpha, \beta, 0) = w = u_1\left(\alpha, \beta, \frac{2K^n}{3}\right)$$

$$u_2(\alpha, \beta, \gamma) = 0$$

$$u_2(0, \beta, \gamma, t) = 0 = u_2\left(\frac{2K^n}{3}, \beta, \gamma, t\right) \quad (14)$$

$$u_2(\alpha, 0, \gamma, t) = 0 = u_2\left(\alpha, \frac{2K^n}{3}, \gamma, t\right)$$

$$u_2(\alpha, \beta, 0, t) = 0 = u_2\left(\alpha, \beta, \frac{2K^n}{3}, t\right)$$

where K^n is the perpendicular distance from the vortex of reference triangle to its opposite side and the trilinear coordinates, a, β, γ of any point are related by

$$a + \beta + \gamma = K^n \quad (15)$$

and $$\nabla^2_{\alpha,\beta,\gamma} = \frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} + \frac{\partial^2}{\partial\gamma^2} - \frac{\partial^2}{\partial\alpha\,\partial\beta} - \frac{\partial^2}{\partial\beta\,\partial\gamma} - \frac{\partial^2}{\partial\gamma\,\partial\alpha}$$

The other physical requirement is that u must be finite.

4. SOLUTION OF THE PROBLEM

To solve the problem, we shall use the following developed integral transform of function of trilinear coordinates :

$$T\Big[f(\alpha, \beta, \gamma) = f^*(r) = \int_0^{K_1}\int_0^{K_2}\int_0^{K_3} f(\alpha, \beta, \gamma)$$
$$\left\{\sin\frac{2r\pi\alpha}{K^n} + \sin\frac{2r\pi\beta}{K^n} + \sin\frac{2r\pi\gamma}{K^n}\right\} d\alpha\, d\beta\, d\gamma \quad (16)$$

where $K_1 = K_2 = K_0 = \frac{q}{p} K^n$, p, q and r being integers.

The transform has the inverse formula

$$f(\alpha, \beta, \gamma) = \sum_{r=0}^{\infty} f^*(r) C_r \left\{\sin\frac{2r\pi\alpha}{K^n} + \sin\frac{2r\pi\beta}{K^n} + \sin\frac{2rn\gamma}{K^n}\right\} \quad (17)$$

where $\frac{1}{C_r} = \frac{3}{2}\left(\frac{q}{P} K^n\right)^3$

with the help of operational property

$$T\Big[(L) f(\alpha, \beta, \gamma) = -\sum_{r=0}^{K_s}\sum_{r=0}^{K_t}\left[\frac{\partial u_r}{\partial x_i} f(\alpha, \beta, \gamma)\right]_0^{K_1} dx_s\, dx_t - \frac{4\pi^2 r^2}{K^{n2}} f^*(r) \quad (18)$$

where $I \neq s$ and $s \neq t$ and dn_i are perpendicular distances to the sides of the reference triangle in the trilinear system of coordinates and the function u_r is

$$u_r = \left\{\sin\frac{2r\pi\alpha}{K^n} + \sin\frac{2r\pi\beta}{K^n} + \sin\frac{2r\pi\gamma}{K^n}\right\} \quad (19)$$

Now solution of Eqn. (11) satisfying the boundary conditions (13) is

$$u_1 = w + \sum_{r=1}^{\infty} \frac{2}{r\pi}\left(\frac{K^n}{2\pi r}\right)\frac{P}{\gamma}\left\{\sin\frac{2r\pi\alpha}{K^n} + \sin\frac{2r\pi\beta}{K^n} + \sin\frac{2r\pi\gamma}{K^n}\right\} \tag{20}$$

This solution can be verified by in Eqn. (11) and using that

$$K^n - 2\alpha = \sum_{r=1}^{\infty} \frac{2K^n}{r\pi}\sin\frac{2r\pi\alpha}{K^n}$$

$$K^n - 2\beta = \sum_{r=1}^{\infty} \frac{2K^n}{r\pi}\sin\frac{2r\pi\beta}{K^n}$$

$$K^n - 2\gamma = \sum_{r=1}^{\infty} \frac{2K^n}{r\pi}\sin\frac{2r\pi\gamma}{K^n}$$

and $\sum_{r=1}^{\infty} \frac{2}{r\pi}\left\{\sin\frac{2r\alpha\pi}{K^n} + \sin\frac{2r\pi\beta}{K^n} + \sin\frac{2r\pi\gamma}{K^n}\right\} = 1.$

Again solution of Eqn. (12) w.r.t. boundary and initial conditions (14), we use Laplace transform for the time variable so that Eqn. (12) becomes,

$$s^2 + \left\{\frac{k}{\tau} + \frac{1}{\tau} + \frac{m\gamma}{\tau}\left(\frac{1}{K'} + M^2\right)\right\}s - \gamma\left(\frac{1}{\tau} + s\right)\nabla^2_{\alpha,\beta,\gamma}\bar{u}_2$$
$$= \left(\frac{1}{\tau} + s\right)\bar{f}(s) \tag{21}$$

Applying the integral transform of trilinear co-ordinates as defined by Eqns. (16) to (21), we eliminate 2 operator from it, so as to obtain

$$\left[s^2 + \left\{\left(\frac{k}{\tau} + \frac{1}{\tau}\right) + \frac{m\gamma}{\tau}\left(\frac{1}{K'} + M^2\right)\right\}s + \gamma\left(\frac{1}{\tau} + s\right)\left(\frac{6r\pi}{K^n}\right)^2\bar{u}_2^*\right.$$
$$= \frac{4K^{n^3}}{27r\pi}\left(\frac{1}{\tau} + s\right)\bar{f}(s) \tag{22}$$

Hence we have

$$\bar{u}_2^* = \frac{4K^{n^3}}{27r}\,\frac{\bar{f}(s)}{(r_1 - r)}\,\frac{\left(\frac{1}{\tau} + r_1\right)}{s - r_1} - \frac{\frac{1}{\tau} + r_2}{s - r_2} \tag{23}$$

where r_1 and r_2 are roots of the equation

$$s^2 + \left\{\frac{k}{\tau} + \frac{1}{\tau} + \frac{m\gamma}{\tau}\left(\frac{1}{K'} + M^2\right) + \gamma\left(\frac{6r\pi}{K^n}\right)^2\right\}s + \gamma\,\frac{1}{\tau}\left(\frac{6r}{K^n}\right)^2 = 0 \tag{24}$$

To get the solution in terms of the original variables a, β, γ and t, we make use of inversion theorem (17) and inverse Laplace transform, we get

$$u_2(\alpha, \beta, \gamma, t) = \sum_{r=1}^{\infty} \frac{4K^{n^3}}{27r(r_1 - r_2)} \frac{2}{3} \left(\frac{3}{2K^n}\right)^3 u_r \cdot T_m \tag{25}$$

where

$$T_m = \left(\frac{1}{\tau} + r_1\right) \int_0^t e^{r_1\lambda} f(t - \lambda)\, d\lambda - \left(\frac{1}{\tau} + r_2\right) \int_0^t e^{r_2\lambda} f(t - \lambda)\, d\lambda \tag{26}$$

Substituting Eqns. (20) and (26) in Eqn. (7), we get

$$u(\alpha, \beta, \gamma, t) = w + \sum_{r=1}^{\infty} \frac{2}{r\pi} \left(\frac{K^n}{2r\pi}\right)^2 \frac{P}{\gamma} u_r + \sum_{r=1}^{\infty} \frac{1}{3r\pi(r_1 - r_2)} u_r T_m \tag{27}$$

Substituting the values of u in Eqn. (1), we have

$$\begin{aligned} v = &\left\{1 + \frac{\tau}{k}\left(\frac{1}{K'} + M^2\right)\right\} \left[w + \sum_{r=1}^{\infty} \frac{2}{r\pi} \left(\frac{K^n}{2r\pi}\right)^2 \frac{P}{\gamma} u_r + \sum_{r=1}^{\infty} \frac{u_r T_m}{3r\pi(r_1 - r_2)}\right] \\ &+ \frac{\tau}{k}\left[\frac{\partial}{\partial t}\left\{\sum_{r=1}^{\infty} \frac{u_r T_m}{3r(r_1 - r_2)}\right\} - f(t) + \gamma \sum_{r=1}^{\infty} \left\{\frac{2P}{r} + \frac{4r T_m}{3K^{n^3}(r_1 - r_2)} u_r\right]\right. \end{aligned} \tag{28}$$

Eqns. (27) and (28) give the general solution for u and v satisfying all the boundary and initial conditions.

5. PARTICULAR CASES

5.1 Flow under an Impulsive Pressure Gradient

Let $f(t) = A\,\delta(t)$

where A is a positive constant and t is the Dirac delta function. Physically this represent the case when the impulsive pressure gradient of magnitude of A is suddenly increased on the fluid at $t = 0^+$

The solution of Eqns. (27) and (28) comes out to

$$u = w = \sum_{r=1}^{\infty} \frac{2}{r\pi} \left(\frac{K^n}{2r}\right) \frac{P}{\gamma} u_r + \sum_{r=1}^{\infty} \frac{A u_r S_m}{3r\pi(r_1 - r_2)} \tag{29}$$

and

$$\begin{aligned} v = &\left\{1 + \frac{\gamma\tau}{k}\left(\frac{1}{K'} + M^2\right)\right\} \left[w + \sum_{r=1}^{\infty} \frac{2}{r\pi} \left(\frac{K^n}{2r\pi}\right)^2 \frac{P}{\gamma} u_r \right. \\ &+ \sum_{r=1}^{\infty} \frac{A u_r S_m}{3r\pi(r_1 - r_2)} + \frac{\tau}{k}\left[\frac{\partial}{\partial t}\left\{\sum_{r=1}^{\infty} \frac{A u_r S_m}{3r\pi(r_1 - r_2)}\right.\right. \end{aligned}$$

$$-f(t) + \gamma \sum_{r=1}^{\infty} \left\{\frac{2P}{r\pi} + \frac{4\pi A_r S_m}{3K^{n^2}(r_1 - r_2)}\right\} u_r \Big] \tag{30}$$

where

$$S_m = \left(\frac{1}{\tau} + r_1\right)e^{r_1 t} - \left(\frac{1}{\tau} + r_1\right)e^{r_2 t}.$$

5.2 Flow under Constant Pressure Gradient

Let $f(t) = A\ H(t)$

where A is constant and $H(t)$ is the heavyside unit step function.

$$u = w + \sum_{r=1}^{\infty} \frac{2}{r}\left(\frac{K^n}{2r\pi}\right)^2 \frac{P}{\gamma} u_r + \sum_{r=1}^{\infty} \frac{A u_r R_m}{3r\pi(r_1 - r_2)} \tag{31}$$

$$v = \left\{1 + \frac{\gamma\tau}{k}\left(\frac{1}{K'} + M^2\right)\right\}\left[w + \sum_{r=1}^{\infty} \frac{2}{r\pi}\left(\frac{K^n}{2r\pi}\right)^2 \frac{P}{\gamma} u_r\right.$$

$$+ \sum_{r=1}^{\infty} \frac{A u_r R_m}{3r\pi(r_1 - r_2)} + \frac{\tau}{k}\left[\frac{\partial}{\partial t}\left\{\sum_{r=1}^{\infty} \frac{A u_r R_m}{3r\pi(r_1 - r_2)}\right\} - f(t)\right.$$

$$\left. + \left\{\gamma \sum_{r=1}^{\infty} \frac{2P}{r\pi\gamma} + \frac{4A_r R_m}{3K^{n^2}(r_1 - r_2)}\right\} u_r\right] \tag{32}$$

Evidently, solutions (29), (30), (31) and (32) satisfy the boundary and initial conditions.

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