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Heat Transfer from a Conducting Hollow Cylinder

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ABSTRACT

An analytical solution has been obtained for the conjugated problem of heat transfer from an infinite vertical hollow cylinder of finite thickness to the surrounding infinite quiescent fluid, following a step change in either heat flux or temperature, at the inner surface of cylinder. The effect of various non-dimensional parameters on the velocity and temperature profiles have been exhibited graphically.

NOMENCLATURE

a'	external radius
b'	internal radius
H	coefficient of heat flux
r'	radial distance
T	absolute temperature
u	axial velocity
ν	kinetic viscosity
ρ	density

Non-dimensional quantities

G	$g\beta a' (T_1 - T_\infty)/\nu^2$
θ	$g\beta a' (T_2 - T_\infty)/\nu^2$
h	Ha'/k_1
K	ratio of conductivities
σ	Prandtl number

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k ratio of diffusivities

K^* $\sqrt{k/K}$

τ $\nu t/a'^2$

u $u'a'/\nu^2$

r'/a'

Subscripts

1 for fluid

2 for solid

INTRODUCTION

Siegel[1] observed that in early stages of non-steady free convection the heat transfer is purely by conduction. 'Initial conduction regime', a period between the transition and the onset of convection has also been studied by Schetz and Eichhorn[2]. Manold and Young[3] simultaneously studied the same problem including additional cases of linear and sinusoidal variation in heat flux. Goldstein and Briggs[4] studied the problem of semi-infinite plate taking into account the heat capacity. Nimbu[5] has recently made a study of pure conduction by similarity analysis and has shown that for a time less than critical value the fluid is unaware of leading edge and temperature and velocity profiles are the same as for doubly infinite plate. Bal Krishan[6,7] has recently studied the conjugated heat transfer problems in free and forced convections by taking implicit boundary conditions at solid-fluid interface. In this paper the conjugated thermal character of the problem has been retained by considering the conduction equation through the finite thickness of vertical hollow cylinder. The study finds its application in natural cooling of chambers that receive heat impulsively at the transient, for example, nuclear reactors, rocket launchers and also in the design of cooling systems.

2. PROBLEM

Consider an infinite vertical hollow cylinder of finite thickness with a' and b' as external and internal radii respectively. Its outer surface is in contact with a quiescent fluid occupying the space $a' \leq r' \leq \infty$. Initially the cylinder and the fluid are at a common temperature T. At time $t = 0$ the inner surface $r' = b'$ undergoes a step change in heat flux (Case (i)) or in temperature (Case (ii)), which are constantly maintained. We determine the temperature and velocity distribution in the solid and fluid regions under the matching conditions at the interface $r' = a'$.

For the problem stated we consider the equations

Momentum equation

$$\rho_1 \frac{\partial u'}{\partial t} = \mu\left(\frac{\partial^2 u'}{\partial r'^2} + \frac{1}{r'}\frac{\partial u'}{\partial r'}\right) + g\rho_1\beta(T_1 - T\infty) \tag{1}$$

Energy equation

$$\rho_1 c_1 \frac{\partial T_1}{\partial t} = K_1\left(\frac{\partial^2 T_1}{\partial r'^2} + \frac{1}{r'}\frac{\partial T_1}{\partial r'}\right) \tag{2}$$

and

Conduction equation

$$\rho_2 c_2 \frac{\partial T_2}{\partial t} = K_2 \left(\frac{\partial^2 T_2}{\partial r'^2} + \frac{1}{r'} \frac{\partial T_2}{\partial r'} \right) \tag{3}$$

with initial conditions

(a) $T_1 = T_2 \quad T_0 \quad u' = 0$

and boundary conditions

(b) $K_1 \frac{\partial T_1}{\partial r'} \quad K_2 \frac{\partial T_2}{\partial r'} \quad \text{at } r' \quad a$

(c) $K_1 \frac{\partial T_1}{\partial r'} \quad H(T_1 \quad T_2) \quad \text{at } r' \quad a'$ (4)

and

(d) $\frac{\partial T_2}{\partial r'} = \quad S_F \text{ (const.) at } r' = b',$ Case (i)

or

(e) $T_2 \quad S_T \text{ (const.) at } r' \quad b'$ Case (ii)

In non-dimensional form, these equations can be written as

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial r^2} + \frac{}{r} \frac{\partial u}{\partial r} + G \tag{5}$$

$$\frac{\partial G}{} = \frac{1}{\sigma} \left(\frac{\partial^2 G}{\partial r^2} + \frac{1}{r} \frac{\partial G}{\partial r} \right) \tag{6}$$

and

$$\frac{\partial \theta}{\partial \tau} \quad \frac{k}{\sigma} \left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} \right) \tag{7}$$

with initial conditions

(a) $G = \theta = u = 0$

and boundary conditions

(b) $\frac{\partial G}{\partial r} = K \frac{\partial \theta}{\partial r} \quad \text{at } r$

(c) $\frac{\partial G}{\partial r} = h(G - \theta) \quad \text{at } r \quad 1$ (8)

and

(d) $\frac{\partial \theta}{\partial r} \quad S_1 \quad \text{(const.) at } r \quad b$ Case (i) or

(e) $\theta = S_2$ (const.) at $r = b$ Case (ii)

where

$r = r'/a'$ and $b = b'/a'$

3. SOLUTION

Applying Laplace transform, Eqns. (5) to (8) become

$$\frac{\partial \bar{u}^2}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} \quad p\bar{u} = \quad G \tag{9}$$

$$\frac{\partial^2 \bar{G}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{G}}{\partial r} \quad p\sigma\bar{G} \quad 0 \tag{10}$$

$$\frac{\partial^2 \bar{\theta}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\theta}}{\partial r} \quad \frac{p\sigma}{k}\bar{\theta} = 0 \tag{11}$$

with boundary conditions

(a) $\frac{\partial \bar{G}}{\partial r} = K\frac{\partial \bar{\theta}}{\partial r}$ at $r = 1$

(b) $\frac{\partial \bar{G}}{\partial r} = h(\bar{G} - \bar{\theta})$ at $r \quad 1$ (12)

(c) $\frac{\partial \bar{\theta}}{\partial r} = -S_1/p$ at $r = b$ Case (i)

or

(d) $\bar{\theta} = S_2/p$ at $r = b$ Case (ii)

Solutions of Eqns. (10) and (1 are

$$\bar{G} = AK_0(q_1 r) \tag{13}$$

and

$$\bar{\theta} = BI_0(q_2 r) + cK_0(q_2 r) \tag{14}$$

where

$q_1^2 = p\sigma, \quad q_2^2 \quad p\sigma/k$

p is Laplace's parameter.

I_0 and K_0 are Bessel functions of second kind and zeroth order.

Case (i) : Step Change in Heat Flux at $r = b$

Evaluating the constants A, B and C from boundary conditions (12) (a,b,c), we have

$$\overline{G} = S_1 hk K_0(q_1 r)/p^2\sigma\Delta \tag{15}$$

and

$$\overline{\theta} = \frac{S_1}{pq_2\Delta}\Big[I_0(q_2 r)\{hK_1(q_2)K_0(q_1) + q_1 K_1(q_2)K_1(q_1) - hK^* K_0(q_2)K_1(q_1)\}$$
$$+ K_0(q_2 r)\{hI_1(q_2)K_0(q_1) + q_1 I_1(q_2)K_1(q_1) + hK^* I_0(q_2)K_1(q_1)\}\Big] \tag{16}$$

where

$$\Delta = K_1(q_2 b)\{hI_1(q_2)K_0(q_1) + q_1 I_1(q_2)K_1(q_1) + hK^* I_0(q_2)K_1(q_1)\}$$
$$- I_1(q_2 b)\{hK_1(q_2)K_0(q_1) + q_1 K_1(q_2)K_1(q_1) - hK^* K_0(q_2)K_1(q_1)\}$$

To get the solutions valid for small values of time, we expand $\overline{G}$ and $\overline{\theta}$ for large values of p by using the assympotic expansions of the bessel functions. Thus we get the expansions of $\overline{G}$ and $\overline{\theta}$ in series in ascending powers of $1/q$. After inverting the series term by term

$$G = \frac{2S_1 h\sqrt{bk}}{\sqrt{r}}\sum_{n=0}^{\infty}\left\{\left(1 - \frac{2nK^*}{K_1^*}\right) F_1\left(r_n, \frac{\tau}{\sigma}\right) + \frac{2nK^*}{K_1^*} F_2\left(r_n, \frac{\tau}{\sigma}\right)\right\} \tag{17}$$

and

$$\theta = \frac{S_1\sqrt{bk}}{\sqrt{r}}\sum_{n=0}^{\infty}\Big[\sum_{\beta=0}^{1}\Big\{2f_1(r_{n\beta}, \tau/\sigma) - r_{n\beta} f_2(r_{n\beta}, \tau/\sigma)$$
$$-\left(2nhK^* - \frac{2(n^2-n)\, h^2K^{*2}}{\ldots}\right) F_1(r_{n\beta}, \tau/\sigma)$$
$$\frac{2(n^2-n)\, h^2K^{*2}}{hK_1^*} F_2(r_{n\beta}, \tau/\sigma)\Big\}$$
$$-\left\{\left(2hK^* - \frac{2nh^2K^{*2}}{hK_1^*}\right)F_1(r_{n1}, \tau/\sigma) + \frac{2nh^2K^{*2}}{hK_1^*} F_2(r_{n1}, \tau/\sigma)\right\}\Big] \tag{18}$$

where

$$r_n = \frac{(2n+1)(1-b)}{\sqrt{k}} + (r-1), \quad r_{n\beta} = \frac{(2n+1)(1-b)}{\sqrt{k}} + \frac{(-1)^\beta(r-1)}{\sqrt{k}}$$

$$f_1(x, \tau/\sigma) = \left(\frac{1}{\pi\sigma\tau}\right)^{\frac{1}{2}} \exp\left(-\frac{x^2\sigma}{4\tau}\right), \quad f_2(x, \tau/\sigma) = erfc\,(x\sqrt{\sigma}/2\sqrt{\tau})$$

$$f_3(x, \tau/\sigma) = \exp\,(hK_1^* x + h^2K_1^{*2}\,\tau/\sigma)\; erfc\left(\frac{x}{2}\sqrt{\frac{\sigma}{\tau}} + hK_1^*\sqrt{\frac{\tau}{\sigma}}\right)$$

$$F_1(x, \tau/\sigma) = \frac{2\tau}{\ldots} f_1 - \frac{1 + hK_1^* x}{\ldots} f_2 + \frac{1}{h^2K_1^{*2}} f_3$$

and

$$F_2(x, \tau/\sigma) = \frac{1}{h^2K_1^{*2}} f_2 \quad \frac{1}{h^2K_1^{*2}} f_1 \quad \frac{hx \;-\; 2h^2K_1^{*2}\tau/\sigma}{}$$

The expression for velocity is given by

$$u \quad \frac{2S_1\sigma\sqrt{bk}}{(\sigma - 1)h^3K_1^{*4}\sqrt{r}} \sum_{n=0}^{\infty} \left(1 \quad \frac{2nK^*}{K_1^*}\right)$$

$$\times \left[f_3(r_{\sigma n}, \tau/\sigma) \quad f_3(r_n, \tau/\sigma) \right.$$

$$\left. \sum_{\beta=0}^{2} \left(2hK_1^* \sqrt{\frac{\tau}{\sigma}}\right)^{\beta} \left\{ i^{\beta} f_2(r_{\sigma n}, \tau/\sigma) \quad i^{\beta} f_2(r_n, \tau/\sigma) \right\}\right] \tag{19}$$

when $\sigma = 1$

$$u = \frac{2S_1\sqrt{bk}\,(1 - r)}{{}^1K_1^{*4}\sqrt{r}} \sum_{n=0}^{\infty} \left(1 \quad \frac{2nK^*}{K_1^*}\right)$$

$$\times \left\{ \frac{hk_1^*}{2} f_3(r_n, \tau) + \frac{1}{4\sqrt{\tau}} \sum_{\beta=1}^{3} (\quad 2hK_1^*\sqrt{\tau})^{\beta} \times b^{\beta-1} f_2(r_n, \tau) \right\} \tag{20}$$

where

$$r_{\sigma,n} = \frac{(2n + 1)(1 - b)}{\sqrt{k}} + \frac{r - 1}{\sqrt{\sigma}}$$

Case (ii) : Step Change in Surface Temperature at $r = b$

Determining the constants A, B and C from boundary conditions (12) (a,b and d) and proceeding as in case (i) the expressions for the temperature distributions in the fluid and the solid regions and the velocity distribution in the fluid are respectively given by

$$G = \frac{2S_2h\sqrt{b}}{\sqrt{r}} \sum_{n=0}^{\infty} (\quad 1)^n \left[\frac{1}{hK_1^*} \{ f_2(r_n, \tau/\sigma) \right.$$

$$\left. f_3(r_n, \tau/\sigma)\} \;-\; 2nhK^* F_2(r_n, \tau/\sigma) \right] \tag{21}$$

$$\theta \quad S_2\sqrt{\frac{b}{r}} \sum_{n=0}^{\infty} (\quad 1)^n \left[\sum_{\beta=0}^{1} \left[f_2(r_{n\beta}, \tau/\sigma) \quad \frac{2nK^*}{K_1^*} \{ f_2(r_{n\beta}, \tau/\sigma) \quad f_3(r_{n\beta}, \tau/\sigma)\} \right.\right.$$

$$\left. + 2(n^2 \quad n)\, h^2K^{*2}F_2(r_{n\beta}, \tau/\sigma) \right] \quad 2\left[\frac{K^*}{K_1^*} \{ f_2(r_{n1}, \tau/\sigma) \quad f_3(r_{n1}, \tau/\sigma)\} \right.$$

$$\left.\left. 2nh^2K^{*2}F_2(r_{n1}, \tau/\sigma) \right] \right] \tag{22}$$

and

$$u = \frac{2S_2\sigma\sqrt{b}}{(\sigma - 1)\, h^2 K_1^{*3}\sqrt{r}} \sum_{n=0}^{\infty} (-1)^{n+1}\Big[f_3(r_{\sigma n}, \tau/\sigma) - f_3(r_n, \tau/\sigma) - \sum_{\beta=0}^{2}\Big(-2hK^*\sqrt{\frac{\tau}{\sigma}}\Big)^{\beta} \times \{i^{\beta} f_2(r_{\sigma n}, \tau/\sigma) - i^{\beta} f_2(r_{n1}, \tau/\sigma)\}\Big]$$

when $\sigma = 1$

$$u = \frac{2S_2(r-1)}{h^2 K_1^{*3}}\sqrt{\frac{b}{r}} \sum_{n=0}^{\infty} (-1)^n \Big\{ \frac{hK_1^{*2}}{2} f_3(r_n, \tau) + \frac{1}{4\sqrt{\tau}} \sum_{\beta=1}^{2} (-2hK_1^*\sqrt{\tau})^{\beta}\, i^{\beta-1} f_2(r_n, \tau)\Big\} \quad (24)$$

4. DISCUSSION

The results obtained reveal that the temperature distribution in solid and fluid regions and the velocity distribution in the fluid depend upon dimensionless parameters γ, σ, τ, h, k and K^*. The variation of flux at inter-face and non-dimensional velocity u with respect to these parameters have been presented in the Figs. 1 to 5.

Figs. 1 and 2 depict respectively for Case (i) and Case (ii), heat flux at interface for different values of τ/σ and h. The heat flux increases in both the cases, as should be expected. In case (i) it rises continuously, in case (ii) it falls gradually after a steep rise in the initial stage after transition, thereby indicating an approach to steady state. Rest of the curves have been drawn for Case (i), curves for Case (ii) behave similarly.

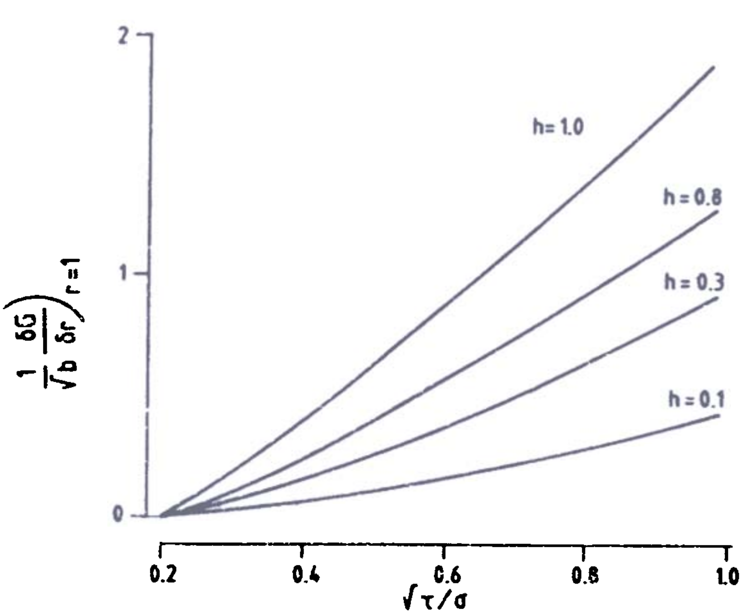

Figure 1 Case (i) – Heat flux at interface $r = 1$, plotted against $\sqrt{\tau/\sigma}$ for different values of h at $K^* = 0.005$, $\sqrt{k} = 2$ and $S_1 = 1$.

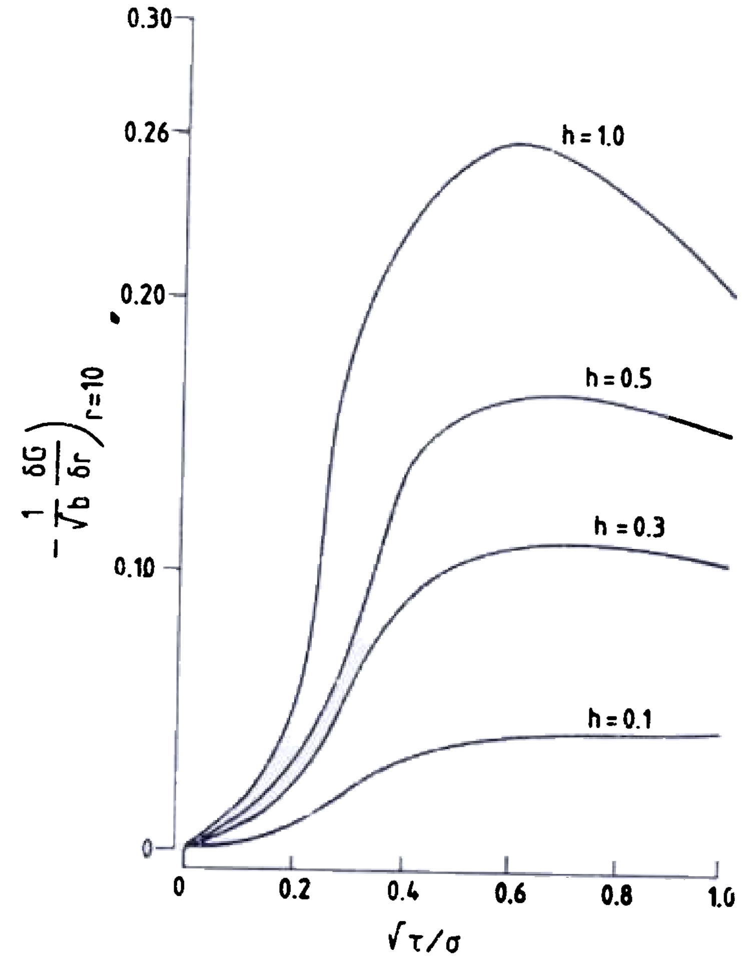

Figure 2. Case (ii) – Heat flux at interface $r = 1$, plotted against $\sqrt{\tau}/\sigma$ for different values of h at $K^* = 0.005$, $\sqrt{k} = 2$ and $S_2 = 1$.

Fig. 3 exhibits velocity profiles which amplify with time and have their maximum near the wall. In Fig. 4 the increase in velocity with increase in k is indicative of increase in heat flux with k. Fig. 5 shows the dependence of heat flux on conjugation parameter

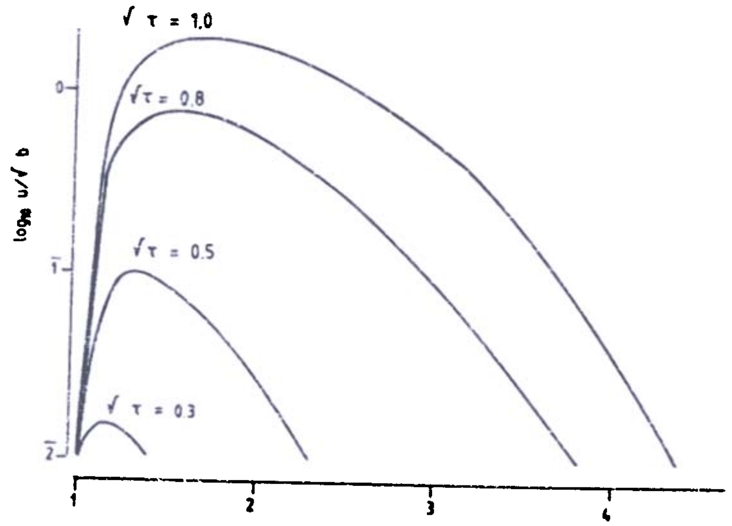

Figure 3. Velocity profiles for different values of τ at $\sqrt{k} = 10$, $h = 1$, $K^* = 0.005$, $\sigma = 1$ and $S_1 = 1$.

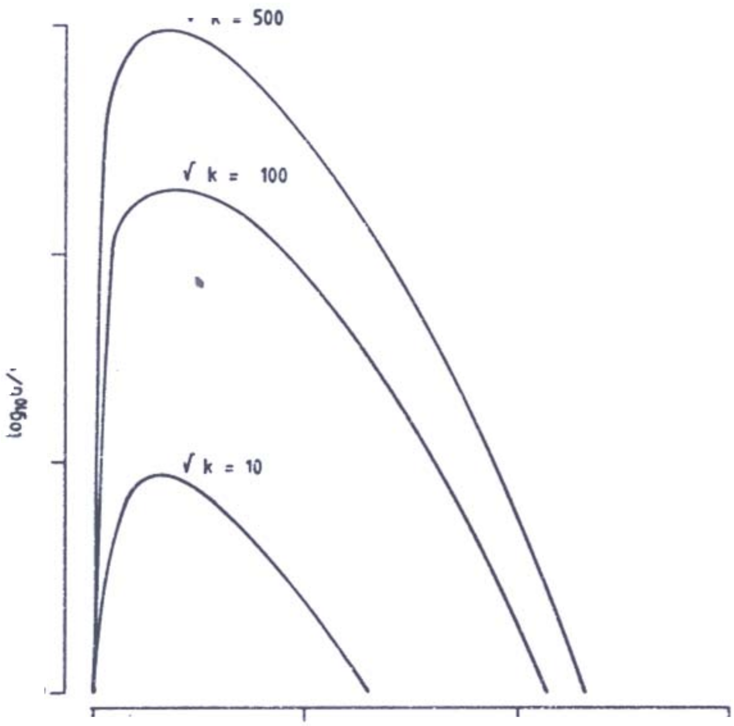

Figure **Velocity profiles** **different values of**
0.5, **= 0.005 and S,**

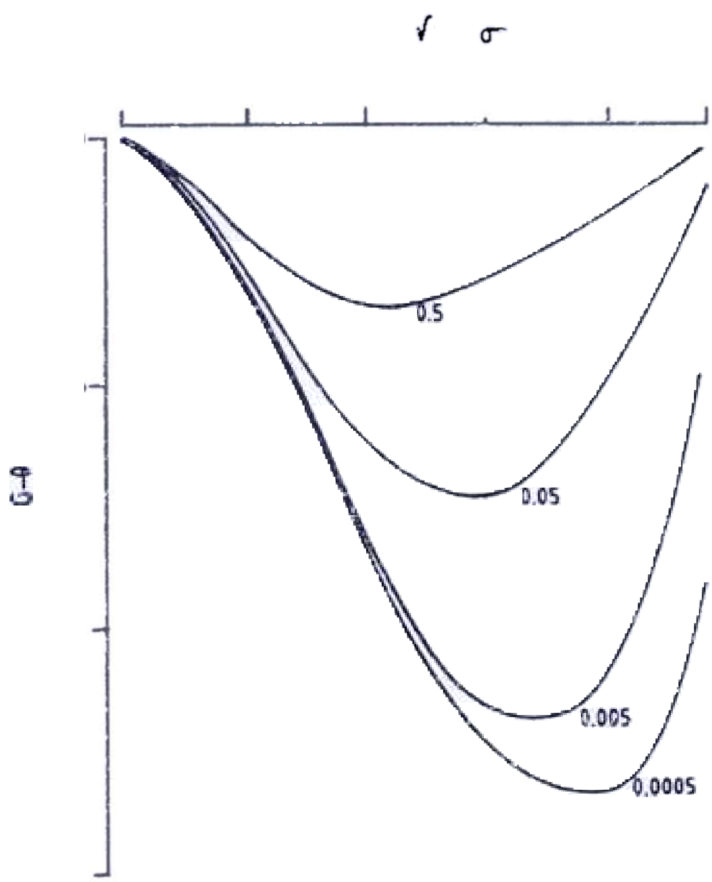

Figure **of** **heat flux** **interface**

K^*, which is symbolic representative of diffusive and conductive properties of both the regions. The heat flux is seen to decrease with the increase in K^*. The effect is more pronounced after $K^* = 0.005$.

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