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Librations of a Quasi-Lagrangian Gyro

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ABSTRACT

The proximate motion of a Lock-Fowler projectile with $q > 0$ is studied via geometrical method due to Copeland. Steady and unsteady motions are classified taking into account conditions at launch.

1. INTRODUCTION

The librations of a quasi-Lagrangian gyro as propounded by Lock and Fowler[1,2] have been geometrically analyzed by Rath and Namboodiri[3]. The qualitative analysis of the librational motions were then analyzed by using the classical method due to Routh[4]. The results obtained did not have a direct reference to the initial conditions of motion. A graphical representation as adopted by Copeland[5] for analysing the motion of a Lagrangian gyro is now adopted here for studying the entire history of motions of the quasi-Lagrangian gyro*. With the aid of Osgood's[6] intrinsic equations extended to the present case, the space curve (signifying the total motion) that could be considered as a trace on a unit sphere is discussed in detail with a set of typical examples. The dependance of the nature of motion on the initial conditions has been shown and how it bifurcates from one type of motion to another when they (initial conditions) are continuously varied has been exemplified. It is also shown in the last section that the Lagrangian gyro is a special case of the present system.

2. EQUATIONS OF MOTION

In analogy with Osgood's equations for a Lagrangian gyroscope one can write the intrinsic equations of motion of a quasi-Lagrangian gyro (in Copeland's form) as follows :

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* A Lock-Fowler projectile (with $q > 0$) is here defined to be quasi-Lagrangian gyro.

$$Bvv' = \mu(\delta) \sin \delta\delta' \quad (1)$$

$$BkV^2 = ANv = -\mu(\delta) \sin^2 \delta\phi' \quad (2)$$

$$A\dot{N} = 0 \quad (3)$$

$$\mu(\delta) = B\Omega^2 (1 - 4qs + 4qs \cos \delta)/4s$$

Here A and B are respectively the axial and transverse moments of inertia of the gyro and N denotes the angular spin about the symmetrical axis. δ and ϕ are the colatitude and longitude of a point P on the axis, where it intersects a unit sphere. If Γ be the trace of P on the unit sphere then k denotes its geodesic curvature at P and v is the instantaneous velocity of P. A dot (.) is used to represent differentiation with respect to time whereas a prime (′) is used to denote differentiation with respect to the arc length σ of Γ. The parameters q and s appearing in the expression for $\mu(\delta)$ are certain constants[3].

From Eqn. (3), we have

$$AN = B\Omega \quad (4)$$

and on integration Eqn. (1) yields

$$v^2 + \Omega^2 (1 - 4\dot{q}s) \cos \delta/2s + q\Omega^2 \cos^2\delta = E \quad (5)$$

where E is the energy integral given by

$$E = V_0^2 + \Omega^2 (1 - 4qs) \cos \delta_0/2s + q\Omega^2\cos^2\delta_0 \quad (6)$$

with δ_0 as the initial value of δ when time $t = t_0$

From Eqns. (5) and (6), we get

$$v^2 = v_0^2 + \Omega^2 (1 - 4qs)(\cos \delta_0 - \cos \delta)/2s + q\Omega^2(\cos^2 \delta_0 - \cos^2 \delta) \quad (7)$$

Since P moves on the unit sphere, we have

$$v^2 = \dot{\delta}^2 + \dot{\phi}^2 \sin^2 \delta \quad (8)$$

when $t = t_0$ Eqn. (8) assumes the value*

$$v_0^2 = \dot{\delta}_0^2 + \dot{\phi}_0^2 \sin^2 \delta_0 \quad (9)$$

The angular momentum integral gives

$$\dot{\phi} \sin^2 \delta + \Omega \cos \delta = F \quad (10)$$

If $\dot{\phi} = \dot{\phi}_0$ when $t = t_0$, F can be expressed as

$$F = \dot{\phi}_0 \sin^2 \delta_0 + \Omega \cos \delta_0 \quad (11)$$

Therefore, setting

$$x = \cos \delta \quad (12)$$

$$a = \Omega^2 (1 - 4qs)/2s \quad (13)$$

$$\beta = q\Omega^2 \quad (14)$$

$$r = F/\Omega \quad (15)$$

* In Eqn. (9), we can assume, without loss of generality that $\dot{\delta}_0 = 0$, implying the non-existence of any initial perturbations, perpendicular to the plane of δ. This implies only a change in the origin of time.

and using Eqns. (6) to (11), we have

$$\dot{x}^2 = -\Omega^2 (r-x)^2 + (E - ax - \beta x^2)(1-x^2) \equiv f(x)$$

and

$$\dot{\phi}(1-x^2) = \Omega(r-x)$$

Using Eqns. (12) to (15) in Eqns. (6) and (11) E and r can be rewritten as

$$E = \dot{\delta}_0^2 + \dot{\phi}_0^2 (1-x_0^2) + ax_0 + \beta x_0^2 \tag{18}$$

$$r = \dot{\phi}_0 (1-x_0^2)/\Omega + x_0 \tag{19}$$

where

$$x_0 = \cos\delta_0 \text{ when } t = t_0$$

It can be seen that the roots x_i (i = 1, 2, 3, 4) of $f(x) = 0$ follow the following distribution :

$$x_1 \leqslant -1 \leqslant x_2 \leqslant x_3 \leqslant +1 \leqslant x_4 \tag{20}$$

Evidently, the real motion of P is bounded between the two levels of limiting motion corresponding to the two real roots x_2 and x_3 (= x_0,* say) that lie in the interval [–1, +1].

3. GRAPHICAL REPRESENTATION OF MOTION

The entire motion is characterised by Eqns. (16) and (17). The parameters that control the motion are x_0, $\dot{\phi}_0$, q, s and Ω. For a given value of q, s, Ω and x_0 it can be seen that the nature of motion changes continuously from one type to another by varying $\dot{\phi}_0$. Thus a change in $\dot{\phi}_0$ would imply a change in a particular type of motion, say, from a wavy structure of Γ to a direct-retrograde type of motion represented by loops. This situation (and any other such situations) can be identified by a certain sets of curves in the $\dot{\phi}_0$–x plane.

In the $\dot{\phi}_0$–x plane, $x = x_i$ (i = 0, 1, 2, 4) are plane curves of which $x = x_0$, in particular, is a straight line. However, owing to the relation (20), we are interested only in the curve $x = x_2$, which is bounded by $x = \pm 1$. The region lying between the curve $x = x_2$ and $x = x_0$ thus, represents the real motion and for a given value of $\dot{\phi}_0$, Γ takes specific shape, which lies entirely within the parallels of latitude given by $x = x_0$ and $x = x_2$. As $x = x_2$ and $x = x_0$ correspond to situations where $\dot{\delta}$ vanishes, similar situations like vanishing of $\dot{\phi}_0$, vanishing of geodisic curvature k and vanishing of the derivative of k (with respect of arclength σ of Γ) are respectively represented by the straight line $x = x_p$, the curve $x = x_g$ and the curve $x = x_k$ (these curves are simple consequences of Eqns. (16), (19) and (2) for $\dot{x}^2 = 0$, $r = 0$, $k = 0$ and $k' = 0$. (cf. Appendix for derivations). These curves are given by

$$x = x_i \equiv f(x)/(x - x_0) = \beta x^3 + (a + \beta x_0)x^2 - (\dot{\phi}_0^2 h + \beta + \Omega^2)\, x$$
$$- \dot{\phi}_0^2 h x_0 + 2\Omega h \dot{\phi}_0 + \Omega^2 x_0 - a - \beta x_0$$

* This being an autonomous system, without loss of generality, we can assume that $f(x_0) = 0$, which implies only a proper choice of the initial conditions. Hence, here after, we presume that $f(x_0) = 0$.

$$x = x_p \equiv \dot{\phi}_0 h/\Omega + x_0$$

$$x = x_g \equiv x_0 + \frac{\Omega}{\beta}\left\{\frac{[\dot{\phi}_0 - (a + 2\beta x_0)]/2\Omega}{[\dot{\phi}_0 + \Omega(a + 2\beta x_0)]/2\beta h}\right\}$$

$$\begin{aligned} x = x_k \equiv\ & (-2\beta h^2/\Omega)\dot{\phi}_0^3 + 2h(3\beta x - \beta x_0 + a)\dot{\phi}_0 \\ & -(4\beta^2 x^2 + 4a\beta x + 2\beta^2 x_0^2 + 2a\beta x_0 + 3a^2/2)\frac{h}{\Omega}\dot{\phi}_0 \\ & + (a + 2\beta x_0)(x_0 - x)(2\beta x - \beta x_0 + a/2) \end{aligned}$$

where

$$h = \ - \dot{x}_0$$

The entering of these curves in the region bounded by $x = x_0$ and $x = x_2$ indicates a qualitative change in the nature of motion. Analysing these curves graphically the nature of Γ can be studied with the help of corresponding rosette diagrams.

Two such typical sets of curves in $\dot{\phi}_0$–x plane are shown in Figs. 1 – 3.

4. CLASSIFICATIONS OF MOTION

The entire set of motion of a quasi-Lagrangian gyro can now be classified with the help of various curves plotted in the $\dot{\phi}_0$–x plane for any given set of initial conditions. In the following paragraphs, we explain, the dependance of motion on the initial conditions, with the help of Figs. 1 and 3. In each case it is discussed, the corresponding rosette diagram (as a specimen) is also given.

4.1 Direct Motion

A region free from $x = x_p$ always corresponds to direct motion as $\dot{\phi}$ does not change its sign at any instant of time. In addition, if such a region contains the curve

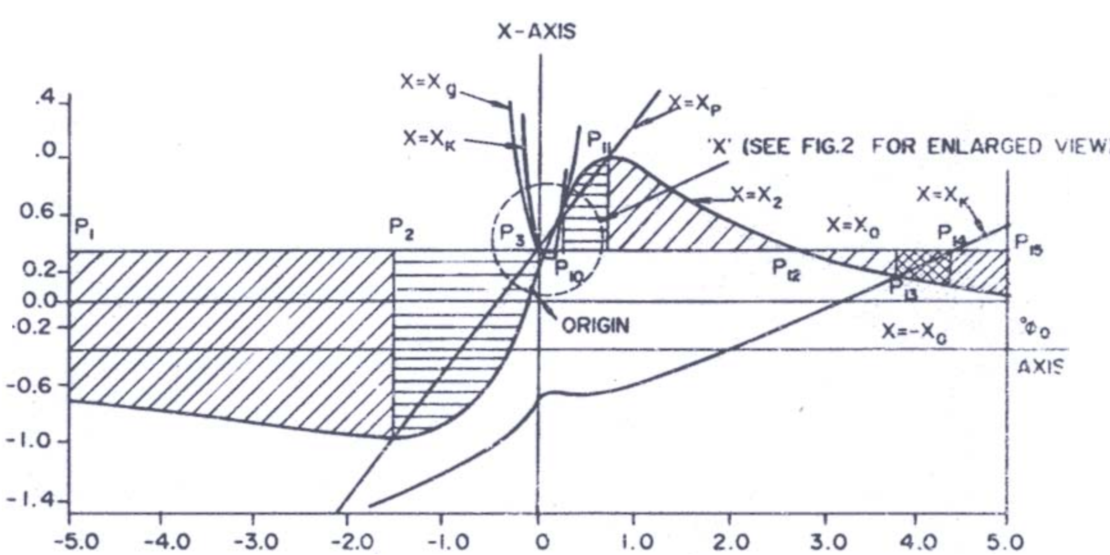


Figure 1. Classification of motion (Ω = 1.0; X_0 = 0.35; δ_0 = 69.56°; a = 0.18; q = 0.05; β = 0.05 and s = 1.79).

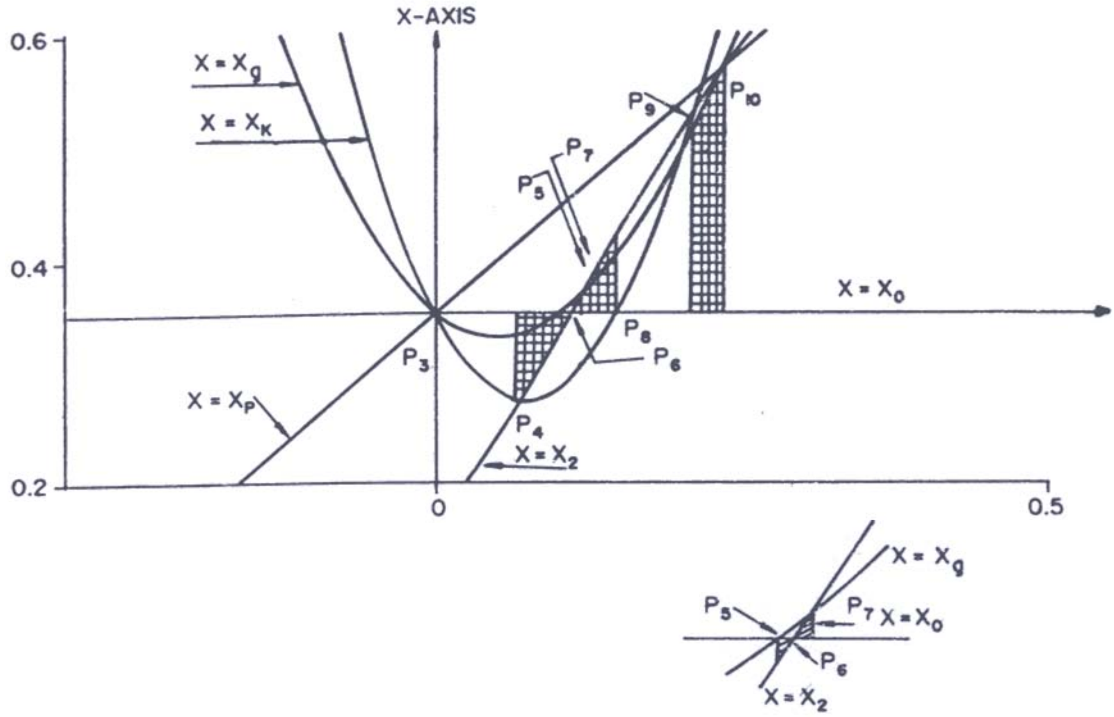


Figure 2. Classification of motion : enlarged view at X of Fig. 1.

$x = x_g$ this would imply an inflection point for Γ and under such situations, the curve Γ will be wavy. On the other hand, if the region is free from both $x = x_p$ and $x = x_g$, it would naturally imply direct motion with non-wavy structure for Γ. If $x = x_k$ enters the region then k will be non-monotonous. Thus in the Figs. 1 and 2 direct and wavy motion is represented by regions between P_3 to P_5 and P_7 to P_{10} and similarly regions between P_1 to P_2, P_5 to P_7 P_{11} to P_{15} represents direct and non-wavy motion, since $x = x_g$ does not enter these regions. As it is clear from the figures that k will be non-monotonous only in the regions P_3 to P_4, P_8 to P_9 and P_{13} and P_{14}. (See Figs. 4 and 5 for rosettes).

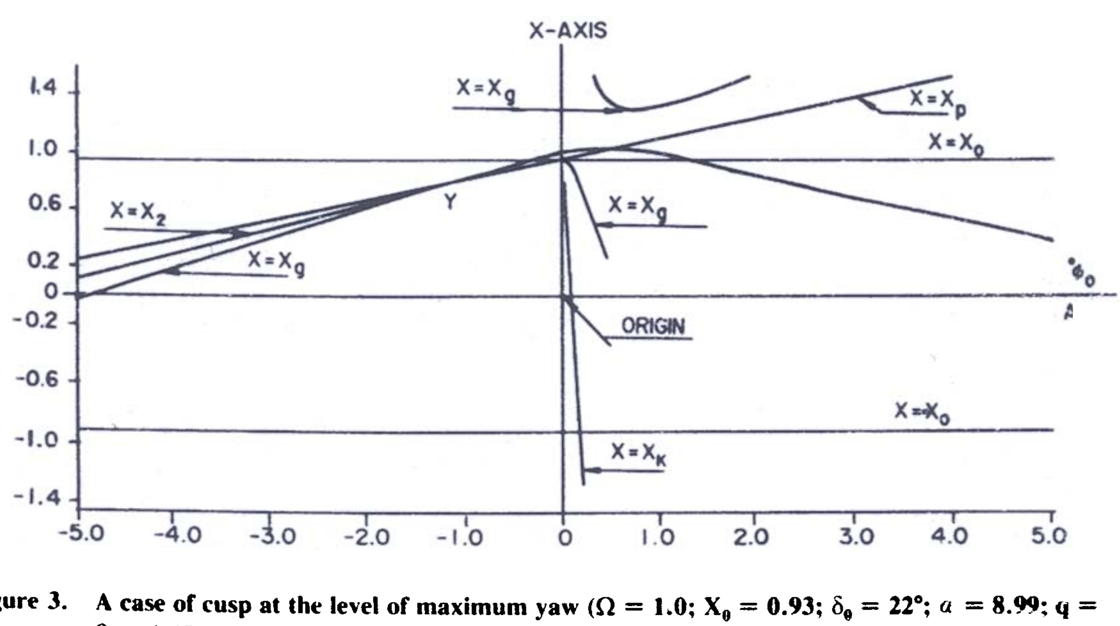


Figure 3. A case of cusp at the level of maximum yaw (Ω = 1.0; X_0 = 0.93; δ_0 = 22°; a = 8.99; q = β = 4.65 and s = 1.67).

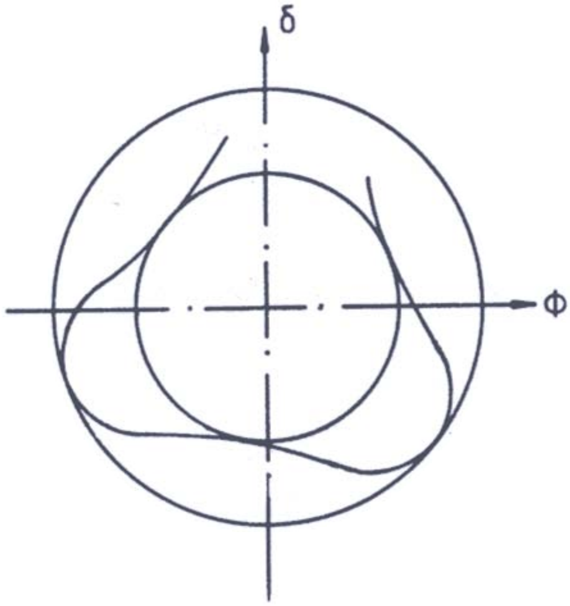


Figure 4. Direct and wavy motion.

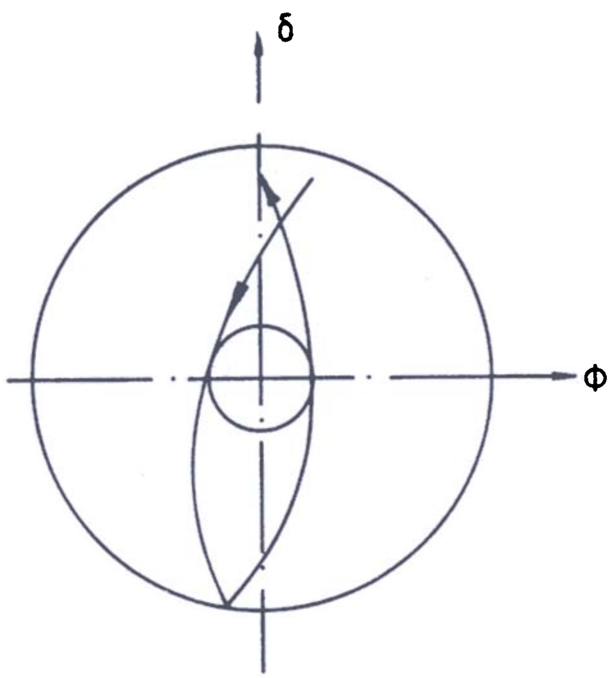


Figure 5. Direct and non-wavy motion.

4.2 Direct and Retrograde Motion (Loops)

A change in the sign of longitudinal velocity of P during its motion from one limiting level to another gives a direct and retrograde motion tracing continuously loops on the unit sphere. Necessarily $\dot{\phi}$ vanishes for a certain value of x in (x_2, x_0). Such a situation is characterised in the $\dot{\phi}_0$–x plane, whenever $x = x_p$ enters the region of real motion. A typical situation is shown in Figs. 1 and 2 for the regions between P_4 to P_5 and P_{10} to P_{11}. Rosette diagram is given in Fig. 6.

4.3 Cuspidal Motion

If at certain points, Γ has got cusps then at such points both the longitudinal and latitudinal velocities must vanish. In the $\dot{\phi}_0$–x plane, this situation arises when the curves $x = x_2$ or the straight line $x = x_0$ intersect $x = x_p$. Points like P_3 and P_{10} in Fig. 1 and 2 refer to such situations. The point P_3 for instance corresponds to initial point of motion. At this point, both the velocities (longitudinal and latitudinal) are zero, as $x = x_p$ intersects $x = x_0$ and hence Γ is a curve with cusps. The first cusp occurs at this initial level and others occuring periodically thereafter on the same level. Although the curve $x = x_g$ passes through this point it does not admit the point of inflection as it corresponds to cuspidal motion. Corresponding rosette is shown in Fig. 7.

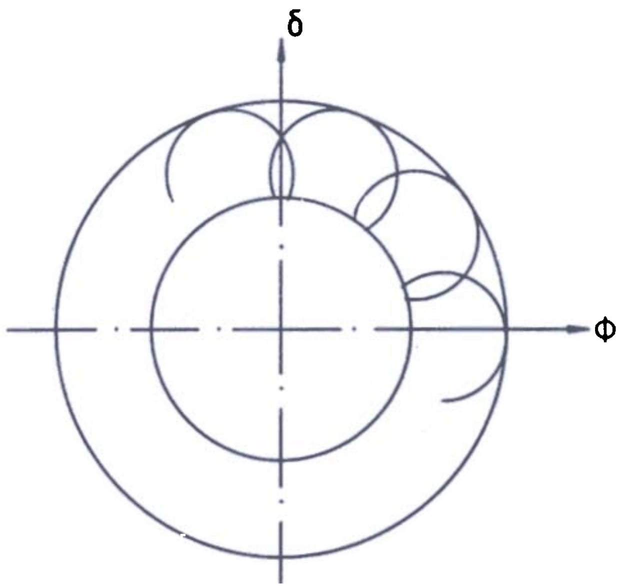


Figure 6. Direct and retrograde motion (loops).

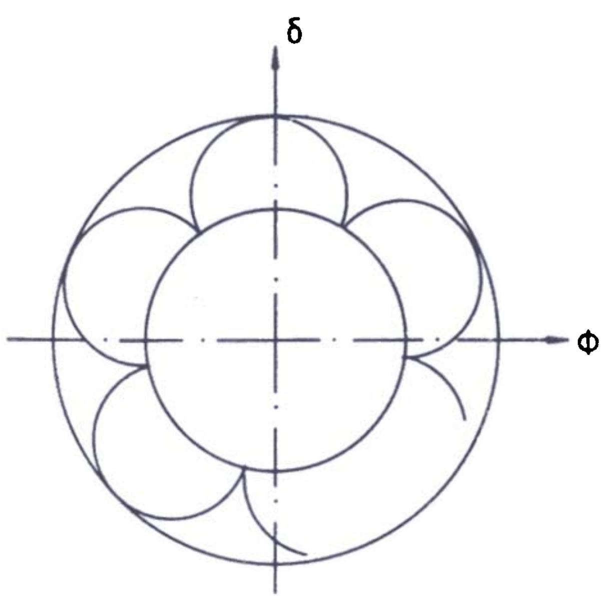


Figure 7. Cuspidal motion.

It may be noted that in all these three cases, the cusps occur at the level of minimum yaw. However, in the case of quasi-Lagrangian gyro, cusps can also occur at the level of maximum yaw. This situation is depicted by the point Y in Fig. 3 and corresponding rosette is shown in Fig. 8.

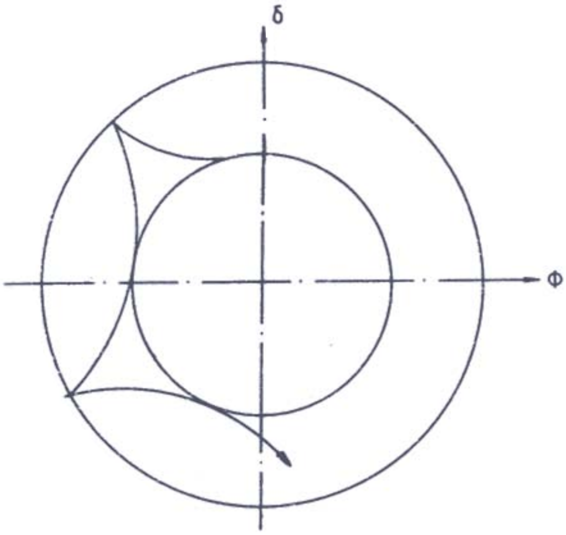


Figure 8. Cusp at the level of maximum yaw.

4.4 Motion Through the Poles

This type of motion usually represented by psuedo-loops occur at $x_2 = \pm 1$ where $x = 0$. However, $x = 0$ is not the consequence of $\dot{\delta} = 0$ but is due to $\delta = 0$ (or π). At $x_2 = \pm 1$, the curve passes through the poles continuously. These situations in $\dot{\phi}_0$–x plane are shown by points P_2 and P_{11} in Fig. 1.

4.5 Steady Precession

The points in the graph Fig. 1 corresponding to steady precessional motion in $\dot{\phi}_0$–x plane are the points where $x = x_0$ and $x = x_2$ intersect. The straight line $x = x_0$ intersects the curve $x = x_2$ in P_6 and P_{12} in Figs. 1 and 2. At these points the parallels of latitude $x = x_0$ and $x = x_2$ coincide to form a single parallel of latitude and reduces to this circle itself thus, exhibiting steady conical motion of the gyroscope.

4.6 Spiral Motion

Motion of this type takes place when the curve $x = x_4$ intersects the curve $x = x_2$ at $x = +1$. The roots $x = x_4$ and $x = x_2$ both become unity. It takes infinite time for P to reach at $x_2 = x_4 = 1$. This situation is depicted in Fig. 9 at $\dot{\phi}_0 = 0.67$. The corresponding rosettee will be as in Fig. 10.

A summary of classification of motion is given in Table 1.

5. SPINNING TOP : (THE LAGRANGIAN GRAVITY GYRO)

It is easily seen that the equations of motion derived in Eqns. (16) and (17) become those of a spinning top when we set $q = 0$. Thus for $q = 0$ Eqns (21) to (24) become

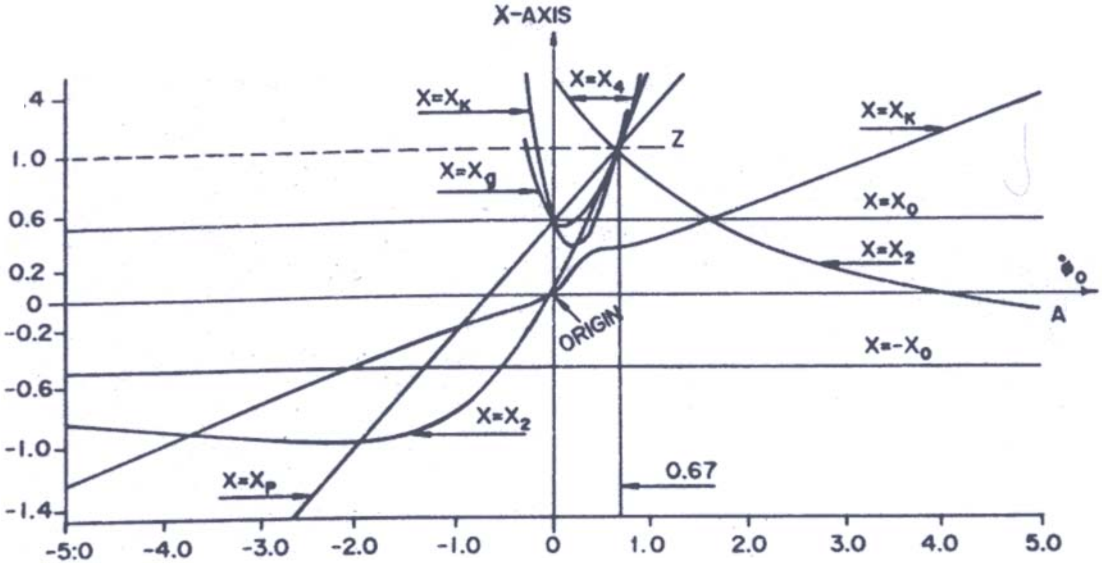


Figure 9. **A case of spiral motion ($\Omega = 1.0$; $X_0 = 0.50$; $\delta_0 = 60°$; $a = 0.27$; $q = 0.27$; $\beta = 0.27$ and $s = 0.63$).**

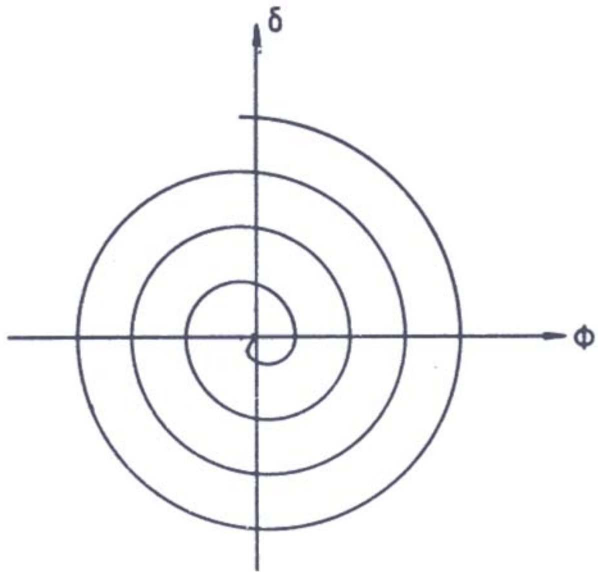


Figure 10. **Spiral motion.**

$$x = x_i \equiv a\ x^2 + (-\dot{\phi}_0^2 h - \Omega^2)x + (-\dot{\phi}_0^2 h x_0 + 2\dot{\phi}_0 h\Omega + \Omega^2 x_0 - a)$$

$$x = x_p \equiv \dot{\phi}_0 h/\Omega + x_0$$

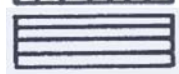
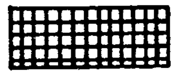
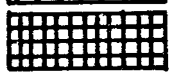
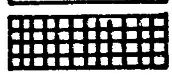
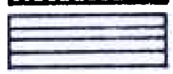
$$x = x_q \equiv (2h\Omega\dot{\phi}_0^2 - ah\dot{\phi}_0 + a\Omega x_0)/a\Omega$$

$$x = x_k \equiv 2h\ \dot{\phi}_0^2 - (3\ ah/2\Omega)\dot{\phi}_0 + a(x_0 - x)/2$$

where

$h = (1 - x_0^2)$

Table 1. Summary of classification of motion

Sr. No.	Region/points in Figs. 1,2,3,9	Nature of motion	Shade used
	P_1-P_2	Direct – nonwavy	
		Direct – retrograde (loops)	
3	P_3-P	Direct – wavy – nonmonotonous k	
4	P_4-P_5	Direct – wavy	
	P_5-P_6	Direct – nonwavy	
6	P_6-P_7	Direct – nonwavy	
7	P_7-P_8	Direct – wavy	
8	P_8-P_9	Direct – wavy – nonmonotonous k	
9	P_9-P_{10}	Direct – wavy	
10	$P_{10}-P_{11}$	Direct – retrograde (loops)	
11	$P_{11}-P_{12}$	Direct – nonwavy	
12	$P_{12}-P_{13}$	Direct – nonwavy	
13	$P_{13}-P_{14}$	Direct – nonwavy – nonmonotonous k	
14	$P_{14}-P_{15}$	Direct – nonwavy	
15	P_3, P_{10}	Cusps at the level of min. yaw	Nil
16	P_6, P_{12}	Steady precession	Nil
17	P_2, P_{11}	Psuedo loops	Nil
18	Y	Cusps at the level of max. yaw	Nil
19	Z	Spiral (asymptotic) motion	Nil

Evidently, these are identical with Eqns. (8), (17), (18) and (21) of Copeland. A typical set of curves when $q = 0$ is given in Fig. 11 for ready reference.

6. CONCLUSION

By adopting Copelands analysis, we have succeeded in classifying the various types of angular motion of a Lock-Fowler projectile. The classification depends on initial conditions chosen contrary to what has been done by Rath and Namboodiri[3] earlier.

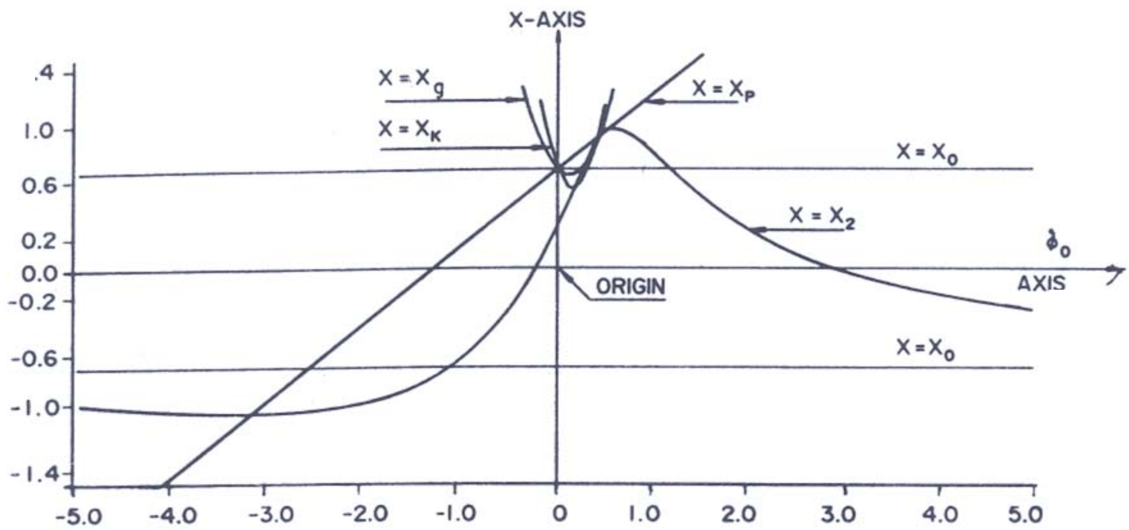


Figure 11. Classification of motion (Lagrangian gravity gyro) (Ω = 1.0; X_θ = 0.68; δ_θ = 47°; a = 0.42; q =0; β = 0.0 and s = 1.20).

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APPENDIX

(1) The Curve : $x = x_g$

The expression for k can be obtained by solving-Eqn. (2) for k; that is

$$k = \left\{ ANv - \mu(\delta) \operatorname{Sin}^2 \delta\phi' \right\} \Big/ Bv^2$$

It follows by substitution, that

$$k = \Omega \left\{ v^2 - \Omega^2 (1 - 4qs + 4qs \cos \delta) \sin^2 \delta\phi'/4s \right\} \Big/ v^3$$

Using relations stated in Eqns. (12) to (19), one gets

$$k = \Omega \left\{ (1 - x_0^2)\, \dot{\phi}_0^2 - (a+2\beta x)\, (1 - x_0^2)\, \dot{\phi}_0/2\Omega \right.$$

$$\left. (a+2\beta x_0)\, (x_0 - x)/2 \right\} \Big/ v^3 \tag{A.3}$$

If $k = 0$ and v is not infinite, Eqn. (A.3) becomes

$$x = \frac{2\Omega\, [\, (a+2\beta x_0)\, x_0/2 + (\dot{\phi}_0 - a/2\Omega)\, h\dot{\phi}_0]}{[\, \Omega\, (a+2\beta x_0) + 2\beta h\, \dot{\phi}_0]}$$

which can be expressed as

$$x = x_g \equiv x_0 + \frac{\Omega}{\beta} \left\{ \frac{[\, \dot{\phi}_0 - (a+2\beta x_0)/2\Omega\,]}{[\, \dot{\phi}_0 + \Omega(a+2\beta x_0)/2\beta h\,]} \right\} \tag{A.4}$$

(2) The Curve : $x = x_k$

An expression for $x = x_k$ can also be derived as follows. Using Eqn. (29), Eqn. (A.3) can be written as :

$$k = \Omega \left\{ \dot{\phi}_0^2 h - (a+2\beta x) h\dot{\phi}_0 2\Omega + (a+2\beta x_0)\, (x_0 - x)/2 \right\} \Big/ v^3 \tag{A.5}$$

Differenting k with respect to arc length σ, we get

$$\frac{v^6}{\Omega} k' = v^3 \left\{ (-\beta h\phi_0/\Omega)\, \frac{dx}{d\sigma} \quad [\, (\, a+2\beta x_0)/2\,]\, \frac{dx}{d\sigma} \right\}$$

$$[\, \dot{\phi}_0^2 h - (a+2\beta x) h\dot{\phi}_0 2\Omega + (a+2\beta x_0)\, (x_0 - x)\,]\, 3v^2\, \frac{dv}{d\sigma} \tag{A.6}$$

Using Eqns (13) and (14) in Eqn. (5), we have

$$v^2 = E - ax - \beta x^2$$

we also have

$$\frac{dx}{d\sigma} = \frac{dx}{dt}\, \frac{dt}{d\sigma} = \frac{\dot{x}}{v}$$

$$3v^2 \frac{dv}{d\sigma} = 3v^2 \frac{dv}{dx}\,\frac{dx}{dt}\,\frac{1}{d\sigma/dt}$$

Using above relations (A.6) becomes :

$$\frac{v^6}{\Omega} k' = -(E - ax - \beta x^2)\,[\,(\beta h/\Omega)\,\dot{\phi}_0 + (a+2\beta x_0)/2\,]\,\dot{x}$$

$$+ 3\left\{ h\dot{\phi}_0^2 - (a+2\beta x)\,(h/2\Omega)\dot{\phi}_0 + (a+2\beta x_0)\,(x_0 - x)/2 \right\} \frac{(a+2\beta x)\dot{x}}{2}$$

Therefore

$$k' = \frac{\Omega \dot{x}}{} \left\{ 3(a+2\beta x)[\,\dot{\phi}_0^2 h - (a+2\beta x)(h/2\Omega)\dot{\phi}_0 + (a+2\beta x_0)(x_0 - x)/2\,] \right.$$

$$\left. -(E - ax - \beta x^2)\,[\,(2\beta h/\Omega)\dot{\phi}_0 + (a+2\beta x_0)\,] \right\}$$

$$E = \dot{\phi}_0^2 h + ax_0 + \beta x_0^2$$

We have

$$k' \equiv \frac{\Omega \dot{x}}{}\ \dot{\psi}(\dot{\phi}_0, x)$$

where

$$\psi(\dot{\phi}_0, x) = (-2\beta h^2/\Omega)\dot{\phi}_0^3 + 2h\,(3\beta x - \beta x_0 + a)\,\dot{\phi}_0^2$$

$$- (4\beta^2 x^2 + 4a\beta x + 2\beta^2 x_0^2 + 2a\beta x_0 + 3a^2/2\,)\,\frac{h}{\Omega}\,\dot{\phi}_0$$

$$+ \frac{3}{2}\,(a+2\beta x_0)\,(a+2\beta x)\,(x_0 - x) - (a+2\beta x_0)\,[\,a\,(x_0 - x) + \beta(x_0^2 - x^2)]$$

k' vanishes, when $\psi(\dot{\phi}_0, x) = 0$. Therefore, we have,

$$x = x_k \equiv (-2\beta h^2/\Omega)\dot{\phi}_0^3 + 2h\,(3\beta x - \beta x_0 + a)\,\dot{\phi}_0^2$$

$$- (4\beta^2 x^2 + 4a\beta x + 2\beta^2 x_0^2 + 2a\beta x_0 + 3a^2/2\,)\,\frac{h}{\Omega}\,\dot{\phi}_0$$

$$+ (a+2\beta x_0)\,(x_0 - x)\,(2\beta x - \beta x_0 + a/2\,) \tag{A.7}$$

which signifies vanishing of k'