Def Sci J, Vol 40, No. 3, July 1990, pp 283-292

Thermal Stresses in an Accreting Medium with Heat Generation

D. Rama Murthy and A.V. Manohara Sharma

Department of Mathematics, Osmania University, Hyderabad-500 007

ABSTRACT

In this paper, the problem of a semi-infinite accreting medium moving with a constant velocity is studied. The heat generation in the medium begins at a constant rate and continues indefinitely. Using the modified heat conduction theory, temperature distribution was determined and the associated thermoelastic problem was solved with Laplace transform technique. The results are evaluated numerically and presented graphically.

1. INTRODUCTION

Development of nuclear sources of energy and the attainment of rocket powered high speed flight has been among the most important technological advances of the last four decades. Thermal stresses arise in many familiar areas, and has been a subject of interest. Severe stresses may be developed in a structure subjected to non-uniform changes in temperature. Aircraft structural designers usually deal with thermal stress problems associated with elevated temperatures in airplane, missile structures, jet engines and nuclear reactors. A knowledge of thermal stresses is of great technological importance in the safe and economical design of aircraft structures.

Thermal stresses arising in many fields play an important role in the determination of material life. Transient thermoelastic problems involve the solution of the Fourier heat conduction equation which predicts infinite heat propagation velocities in solids, which is physically inadmissible. To remove this paradox many investigators like Chester[1] , Boley[2], Morse and Feshbach[3], Tisza[4] worked for the modification of the governing heat conduction equation. Here modified heat conduction equation

Received 31 October 1988, revised 26 December 1988.

suggested by Morse and Feshbach[3], has been taken up to solve a thermoelastic problem of an accreting semi-infinite medium. We know that earth was formed by the accretion of dust cloud hypothesis[5,6]. Here a homogeneous semi-infinite medium moving with constant velocity v, in positive x-direction has been considered. Material is supplied to the surface $x = 0$ of the medium at constant temperature T_0 with constant rate. Initially, the generation of heat begins throughout the medium at the rate A cal/cm^3 s. As the time passes, the surface $x=0$ continues to be held at the constant temperature T_0 and the uniform distribution of heat sources is contained by the material accreting at $x=0$.

As the spherical case presents some mathematical difficulties, a simple mathematical model was considered with the hope that the mathematical solution given here may be of interest to and aid others having related problems involving more physical conditions.

2. FORMULATION AND SOLUTION OF THE PROBLEM

The governing modified heat conduction equation for the above described problem for the one dimensional case is

$$\frac{1}{C^2}\frac{\partial^2 T}{\partial t^2} + \frac{1}{h}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \quad v\frac{\partial T}{\partial x} + \frac{A}{\rho C_v} \tag{1}$$

where h, ρ, C_v and C are the thermal diffusivity, density, specific heat at constant deformation and the velocity of heat propagation considered to be finite here, and are taken to be constants along with v and A.

The initial and boundary conditions are given by

$$\left.\begin{aligned} \frac{\partial T}{\partial t}(x, t) &= 0 \\ T(x, t) &= T_0 \end{aligned}\right| \quad \text{at } t = 0, x > 0 \tag{2}$$

$$T(x, t) = T_0 \quad \text{at} \quad x = 0, \quad t > 0$$

which means that the time rate of temperature is initially zero and is necessary to solve hyperbolic equation and the regularity boundary condition which says that the temperature must be finite at large distance is given by

$$T(x, t) \rightarrow \frac{B}{4}\left[\exp\left(\frac{-C^2 t}{h}\right) + \frac{C^2 t}{h} \qquad \right] \text{as } x \rightarrow \infty \tag{3}$$

which was derived by taking the inverse of $\bar{\theta}\,(z,p)$ in Eqn. (11) as $z \rightarrow \infty$ and using Eqn. (4) where

$$B = \frac{4Ah^2}{CT_0 \rho C_v}$$

Introducing the following non-dimensional variables for distance, time and temperature respectively as

$$z = \frac{C}{2h}x; \quad \tau = \frac{C^2}{2h}t; \text{ and } \theta = \frac{T - T_0}{T_0} \tag{4}$$

into Eqns. (1) to (3), we get

$$\frac{\partial^2\theta}{\partial\tau^2} + 2\frac{\partial\theta}{\partial\tau} \quad \frac{\partial^2\theta}{\partial z^2} \quad \frac{2hv}{C}\frac{\partial\theta}{\partial z} + B \tag{5}$$

with

$$\theta(0, \tau) \quad 0 \quad \theta(z, 0); \frac{\partial\theta}{\partial\tau}(z, 0) = 0 \tag{6}$$

and

$$\theta(z, \tau) \to \frac{B}{4}(2\tau + \exp(-2\tau) - 1) \text{ as } z \to \infty \tag{7}$$

Applying Laplace transform to Eqns. (5) to (7), we get

$$\frac{d^2\bar{\theta}}{dz^2} \quad \frac{2hv}{C}\frac{d\bar{\theta}}{dz} - (p^2 + 2p)\bar{\theta} \quad \frac{B}{p} \tag{8}$$

with

$$\bar{\theta}(0, p) \quad 0 \tag{9}$$

$$\bar{\theta}(z, p) \to Bp^{-2}(p + 2)^{-1} \text{ as } z \to \infty$$

where $\bar{\theta}(z, p)$ is the Laplace transform of the temperature $\theta(z, \tau)$ and p is transform parameter.

Using Eqns. (8) to (10), we get the solution for temperature in transform domain as

$$\bar{\theta}(z, p) = -B[\exp(m_2 z) - 1]p^{-2}(p + 2)^{-1} \tag{11}$$

where

$$m_{1,2} \quad \frac{hv}{C} \pm [(p + 1)^2 \quad g^2]^{1/2}$$

and

$$g^2 \quad 1 \quad \left(\frac{hv}{C}\right)^2$$

Inverting[7], we get the expression for non-dimensional temperature as

$$\theta(z, \tau) = -B\exp\left(\frac{hv}{C}z\right)\int_0^\tau F_1(z, \tau - w)\, F_2(z, w)\, dw$$
$$+ \frac{1}{2}B\exp\left(\left(\frac{hv}{C} - 1\right)z\right)\int_0^\tau [\exp(2(w - \tau) - 1)]\, \eta(w \quad z)dw$$
$$+ BF_1(z, \tau)$$

where

$$F_1(z, \tau) = \frac{1}{4}[\exp(-2\tau) + 2\tau - 1];$$

$$F_2(z, \tau) = zg \exp(-\tau)\frac{I_1[g(\tau^2 - z^2)^{1/2}]}{(\tau^2 - z^2)^{1/2}}$$

where I_1 (w) is the modified Bessel's function of first order and first kind and η (w) is unit step function.

3. FORMULATION AND SOLUTION OF ASSOCIATED THERMOELASTIC PROBLEM

The governing equation, the initial and boundary conditions for the determination of only non-vanishing thermal stress σ_{xx} are given by

$$C_1^2 \frac{\partial^2\sigma_{xx}}{\partial x^2} - \frac{\partial^2\sigma_{xx}}{\partial t^2} = m\frac{\partial^2 T}{\partial t^2}$$

where $C_1^2 = (\lambda+2\mu)/\rho$ is the longitudinal velocity of elastic wave and λ and μ are Lame's constants and $m = (3\lambda + 2\mu)a$, a is the coefficient of linear thermal expansion. The initial and boundary conditions are taken as

$$\sigma_{xx}(x, 0) = 0, \frac{\partial\sigma_{xx}}{\partial t}(x, 0) = 0, \sigma_{xx}(0, t) = 0$$

The regularity boundary condition is

$$\sigma_{xx}(x, t) \to -\frac{B}{4}\left[\exp\left(\frac{C^2 t}{h}\right) + \frac{C^2 t}{h} \quad \right] \text{as } x \to \infty$$

Using Eqn. (4), Eqns. (13) to (15) are transformed to

$$\frac{\partial^2\sigma}{\partial z^2} - a^2\frac{\partial^2\sigma}{\partial\tau^2} = a^2\frac{\partial^2\theta}{\partial\tau^2}$$

with

$$\sigma(z, 0) = 0 = \frac{\partial\sigma}{\partial\tau}(z, 0), \sigma(0, \tau) = 0$$

and

$$\sigma(z, \tau) \to -\frac{B}{4}[\exp(-2\tau) + 2\tau - 1] \text{ as } z \to \infty$$

where

$$\sigma = \frac{1}{mT_0}\sigma_{xx}, a^2 = \frac{C^2}{C_1^2}$$

Applying Laplace transform to Eqn. (16) and using Eqns. (13) to (15) the solution for stress in transformed domain is given as

$$\bar{\sigma}(z, p) = B[[p^{-2} (\exp(-apz) - 1] + a^2 [\exp(-apz) - \exp(m_2 z)][(m_2^2 - a^2p^2)^{-1}] (p + 2)^{-1}$$

Using complex inversion theorem of Laplace transform, we get the non-dimensional stress as

$$\sigma(z, \tau) = \frac{B}{2}\int_0^{\tau} (1 - \exp(-2w))\, \eta(w - az)\, dw - \frac{1}{4}\exp(-2(\tau - az)) + a^2\left[\frac{F(0)}{F_2(0)} + \frac{F(w_1)}{F_1(w_1)} + \frac{F(w_2)}{F_1(w_2)}\right]$$

where

$$w_{1,2} = \frac{2}{a^2 - 1}\left(1 \pm \frac{ahv}{C}\right)$$

are the poles of the integrand taken for the inverse Laplace transform by complex integration, and

$$F(\xi) = \exp(-az\xi) - \exp(m_2\xi)$$

$$F_1(\xi) = 3(1 - a^2)\xi^2 + 4(1 - a^2)\xi + 2\left(2 + \frac{h^2v^2}{C^2}\right) \frac{2hv}{C}\left[\frac{(\xi + 1)(2\xi + 3) - g^2}{((\xi + 1)^2 - g^2)^{1/2}}\right]$$

$$F_2(\xi) = 6(1 - a^2)\xi + 4(1 - a^2) - \frac{2hv}{C}\left[\frac{2(\xi + 1)^3 - g^2(5\xi + 6)}{((\xi + 1)^2 - g^2)^{3/2}}\right]$$

and ξ is a complex number whose real part is p.

Here as a particular case where a medium has been considered for which $\frac{hv}{C} = 1$, then $m_2 = -p$ and the expressions for temperature and stress are given by

$$\theta(z, \tau) = \frac{B}{2}\int_0^{\tau} [\exp(-2(\tau - w)) - 1]\eta(w - z)\, dw + \frac{B}{4}[\exp(-2\tau) + 2\tau - 1]$$

and

$$\sigma(z, \tau) = \frac{B}{2(1-a^2)} \int_0^{\tau} [1 - \exp(-2(\tau - w))]\eta(w - az)\, dw - \frac{a^2 B}{2(1-a^2)} \int_0^{\tau} [1 - \exp(-2(\tau - w))]\eta(w - z)\, dw + BF_1(z, \tau) \quad (26)$$

The above results for temperature and stress are evaluated numerically.

4. CONCLUSION

The expressions for non-dimensional temperature distribution and stress distribution Eqn. (12) and (20) respectively were obtained and were evaluated numerically by considering a medium for which $hv/C=1$, for various values of z and τ. In Fig. 1, the variation of temperature (θ/B) with distance (z) is given for $\tau = 1$, 2 and 3. From Fig. 1, it can be noticed that as time goes on, the temperature goes on increasing. This is true because the generation of heat begins throughout the

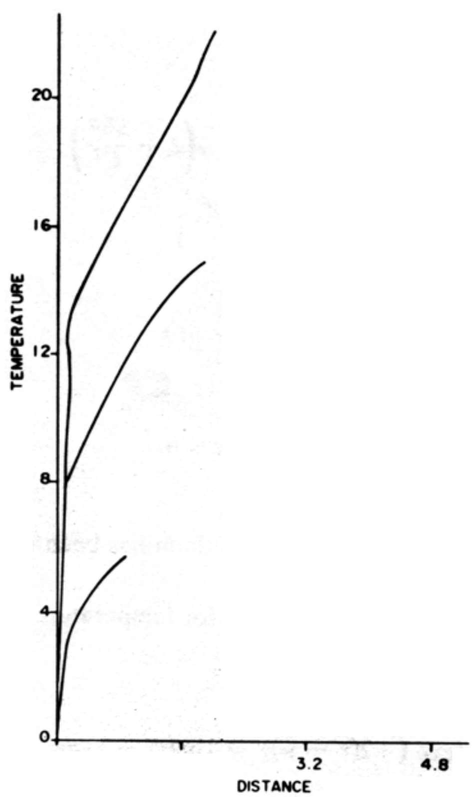

Figure 1. Variation of temperature with distance.

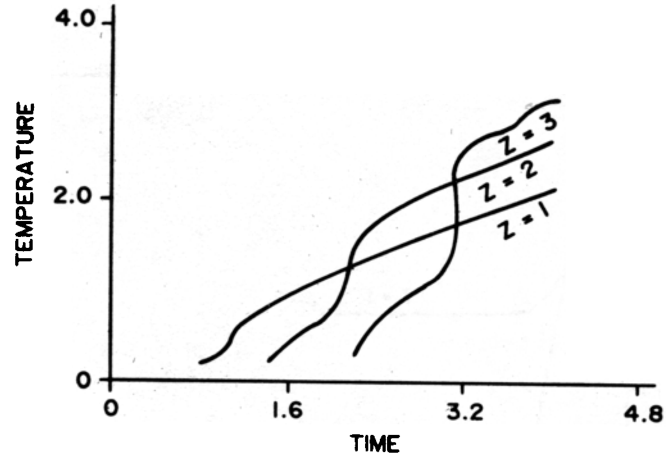

Figure 2. Variation of temperature with time.

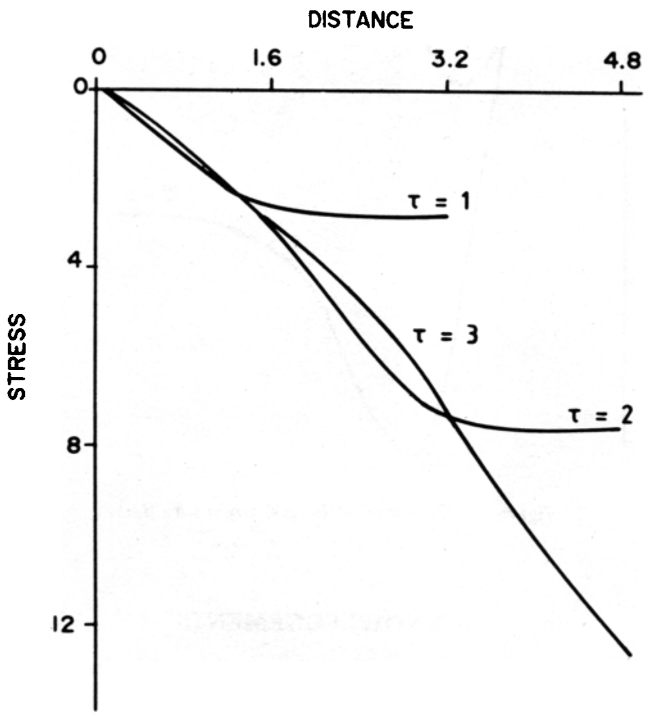

Figure 3. Variation of thermal stress with distance.

semi-infinite medium at a constant rate, and this heat generation continues. In Fig. 2 the variation of the temperature (θ/B) with non-dimensional time, at positions $z = 1, 2$ and 3 is given. The variation of thermal stress (σ/B) with distance and time are given in Figs. 3 and 4 respectively. It is noticed that the stresses are compressive in nature which agrees with the physical nature of the problem.

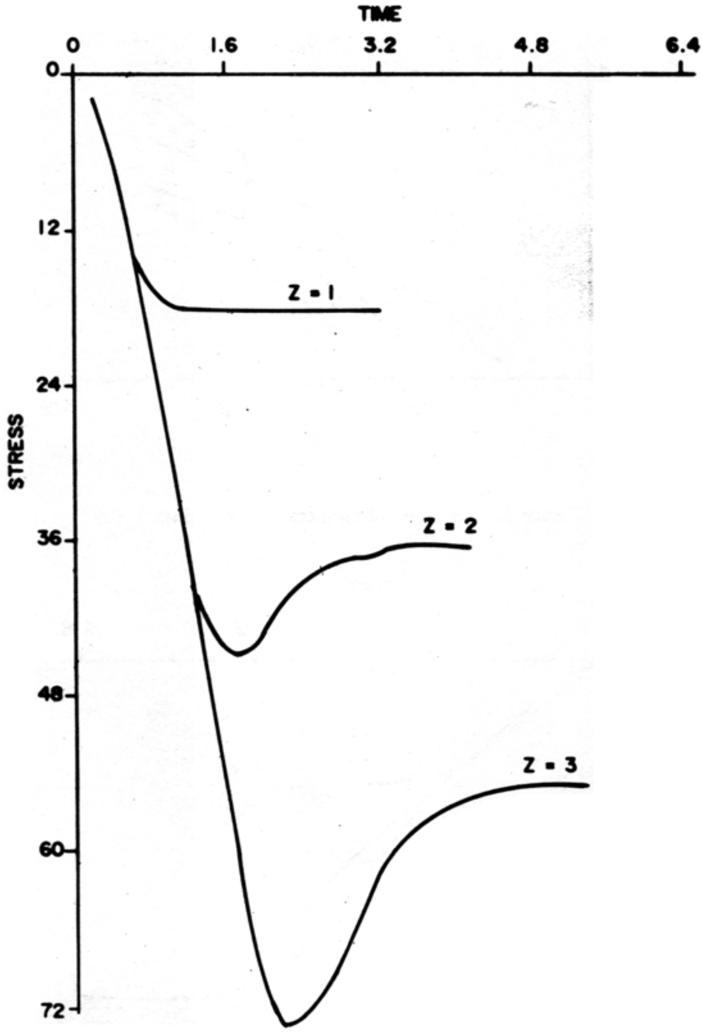

Figure 4. Variation of thermal stress with time.

ACKNOWLEDGEMENTS

The second author thanks Council of Scientific and Industrial Research, New Delhi for the financial assistance.

REFERENCES

Chester, M., *Phy. Rev. Letters,* **131**(1963), 2013.

2. Boley, B.A., *Quart. Appl. Math.,* **19**(1962), 273-284.

3. Morse, P.M. & Feshbach, H., Methods of Theoretical Physics, (McGraw-Hill New York), 1953, p. 865.

4. Tisza, L., *J. Phys. Radium,* (Paris) **1** (1940), 350.
5. Whipple, F.L., *Scientific American,* **178** (1948), 34.
6. Benfield, A.E., *Quart. J. Math.,* **7** (1950), 436-439.
7. Bateman's Manuscript Project : Tables of Integral Transforms, Vol. 1, (McGraw-Hill, New York), 1954.