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Shooting Method for Third Order Simultaneous Ordinary Differential Equations with Application to Magnetohydrodynamic Boundary Layer

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ABSTRACT

An algorithm based on the shooting method has been developed for the solution of a two-point boundary value problem consisting of a system of third order simultaneous ordinary differential equations. The Falkner-Skan equations for electrically conducting viscous fluid with applied magnetic field has been solved by using this algorithm for various values of the wedge angle and magnetic parameters. The shooting method seems to be well convergent for a system as our results are in good agreement with those obtained by other methods. It is observed that both viscous boundary layer and magnetic boundary layer decrease while velocity as well as magnetic field increase with the increase of the wedge angle.

1. INTRODUCTION

The boundary layer problem of the flow past a semi-infinite plate against a pressure gradient was first studied by Falkner and Skan[1]. If the flow outside the boundary layer is parallel to the plate, and is given by $U=kx^m$, where k and m are constants, then this problem reduces to the solution of the following differential equation :

$$F''' = FF'' = \beta(F'^2 - 1) \tag{1}$$

with boundary conditions

$$F(0) = F'(0) = 0; F'(\infty) \to \quad \text{as} \quad \xi \to \infty \tag{2}$$

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where $\beta = 2m/(m+1)$, ξ is a non-dimensional variable perpendicular to the plate, $F(\xi)$ represents the stream function and a prime denotes differentiation with respect to ξ. The solution of this equation was studied by Hartree[2] and Stewartson[3].

Davies[4] extended this problem by considering that the liquid in the neighbourhood of the plate is electrically conducting, the plate is unmagnetized and there is a magnetic field which is parallel to the plate at a large distance from it. He has studied the case when the adverse magnetodynamic pressure gradient is given by cx^n where c and n are constants.

In this paper an algorithm has been developed to solve a boundary value problem consisting of a pair of third order simultaneous ordinary differential equations based on the shooting method[5,6]. The algorithm is applied to solve the equations obtained by Davies[4].

2. THE EQUATION OF MOTION

Consider the flow of a viscous, electrically conducting fluid past a semi-infinite plate given by $y = 0$, $0 < x < \infty$. The liquid flow U and the magnetic field H_0 at a distance from the plate are both assumed to be parallel to the plate so that the problem is the magneto-hydrodynamic version of the problem discussed by Hartree[2] and Stewartson[3]. If (u,v), (H_1, H_2) are velocity and magnetic components in the x and y directions, then the equations governing the flow in the boundary layer region are given[4] by

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \quad -\frac{\partial}{\partial x}\left(\frac{p_\infty}{\rho} + \frac{\mu H_0^2}{8\pi\rho}\right) + \nu\frac{\partial^2 u}{\partial y^2} + \frac{\mu}{4\pi\rho}\left(H_1\frac{\partial H_1}{\partial x} + H_2\frac{\partial H_1}{\partial y}\right) \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4}$$

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_1}{\partial y} = 0 \tag{5}$$

$$u\frac{\partial H_1}{\partial x} + v\frac{\partial H_1}{\partial y} - H_1\frac{\partial u}{\partial x} - H_2\frac{\partial u}{\partial y} \quad \eta\frac{\partial^2 H_1}{\partial y^2} \tag{6}$$

where μ, ν, ρ and $\eta = (4\pi\ \mu\ \sigma)^{-1}$ are respectively the magnetic permeability, kinematic viscosity, electrical conductivity and magnetic diffusivity. The first term in the right hand side of Eqn. (3) is magnetodynamic pressure gradient which is prescribed outside the boundary layer and remains unchanged in form within the boundary layer. Assuming the pressure gradient to be of the form cx^n, Eqn. (3) becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \quad cx^n + \nu\frac{\partial^2 u}{\partial y^2} + \frac{\mu}{4\pi\rho}\left(H_1\frac{\partial H_1}{\partial x} + H_2\frac{\partial H_1}{\partial y}\right) \tag{7}$$

Introducing two functions ϕ and ψ such that

$$u = \frac{\partial \phi}{\partial y}, \qquad v = \frac{\partial \phi}{\partial x} \tag{8}$$

$$-\frac{\partial \psi}{\partial y}, \qquad H_2 \quad \frac{\partial \psi}{\partial x} \tag{9}$$

Eqns. (4) and (5) are satisfied identically. Looking solutions for ϕ and ψ of the form

$$\phi = - KUx^{[(n+3)/4]} F(\xi) \tag{10}$$

$$\psi = -RHx^{[(n+3)/4]} G(\xi) \tag{11}$$

where

$$\xi \quad k^{-1}yx^{(n-1)/4} \tag{12}$$

$$K = \left[\frac{4\nu}{U(n+3)}\right]^{1/2} \tag{13}$$

and U, H and K are constants, then

$$u \quad Ux^{[(n+1)/2]}F' \tag{14}$$

$$v = -\frac{1}{4}kUx^{[(n-1)/2]}[(n+3)F + (n-1)\xi F'] \tag{15}$$

$$H_1 = H_0x^{[(n+1)/2]}G' \tag{16}$$

$$\frac{1}{4}kH_0x^{[(n-1)/4]}[(n \quad 3)G + (n-1)\xi G \tag{17}$$

Substituting these in Eqn. (7) we get,

$$\frac{1}{4}(n+3)U^2[F''' + FF''] \qquad \frac{1}{2}U^2(n+1)F'^2$$
$$-C + \frac{\mu}{4\pi\rho}\left[\frac{n+1}{2}G'^2 - \frac{1}{4}(n+3)GG''\right] \tag{18}$$

At $y = \infty$, the magnetohydrodynamic pressure gradient is given by

$$-\frac{\partial}{\partial x}\left(\frac{p_\infty}{\rho} + \frac{\mu H_0^2}{8\pi\rho}\right) = U\frac{dU}{dx} \qquad \frac{\mu}{4\pi\rho}H_0\frac{dH_0}{dx} \tag{19}$$

Taking $F'(\infty) = 1$, $G'(\infty) =$, then c is given by

$$c = \frac{n+1}{2}\left[\frac{\mu H_0^2}{4\pi\rho} \qquad \right] \quad \frac{n+1}{2}U^2(S-1) \tag{20}$$

and so Eqn. (18) can be written as

$$FF'' + \beta(1 - F'^2) = S[GG'' + \beta(1 - G'^2)] \tag{21}$$

where

$$\beta = \frac{2(n+1)}{(n+3)}$$

and

$$S = \mu H_0^2/4\pi U^2 \rho$$

which is the ratio of the magnetic energy and kinetic energy in the uniform stream. The number S can be represented in terms of Hartman number M as

$$\frac{M^2}{\qquad}$$

where

$$M = \mu H_0 L(\sigma/\rho\nu)^{1/2}$$

$$\frac{UL}{\nu} \text{ (Reynolds number)}$$

$$Rm = 4\pi UL\mu\sigma = UL/\eta \text{ (magnetic Reynolds number)}$$

The numerical value of S is always taken as less than 1, as $S = 1$ the flow will be brought to rest by electromagnetic blocking (see Davis[4]) and there will be no flow when $S \geqslant 1$. This was checked by calculations and hence it has been taken that $S < 1$ in the paper.

Then the Eqn.(6) becomes

$$\lambda G''' + FC'' - GF'' = 0 \tag{22}$$

where $\lambda = \eta/\nu$. The boundary conditions for Eqns. (21) and (22) are

$$= 0, \quad F'(0) = 0, \quad F'(\infty) = 1 \tag{23}$$

$$G(0) = 0, \quad G'(0) = 0, \quad G'(\infty) = 1 \tag{24}$$

The intension in this paper is to solve the system of Eqns. (21) and (22) with boundary conditions in Eqns. (23) and (24) completely for various values of the parameters β, λ and S.

3. METHOD OF SOLUTIONS

Here, to solve the boundary value problem (Eqns (21) and (22)) the shooting method was used. In this method the boundary value problem is converted to an initial value problem by estimating the missing initial conditions to a desired degree of accuracy by an iterative scheme.

Let us consider a general system of third order simultaneous ordinary differential equations

$$y' \quad = f(x, y, y', y'', z, z', z' \tag{25}$$

$$z''' = g(x, y, y', y'', z, z'z'') \tag{26}$$

with boundary conditions

$$\left.\begin{array}{l} x \quad a;\ y = y_a,\ y' \quad y'_a,\ z = z_a,\ z' = z'_a \\ x = b;\ y = y_b,\ z = z_b \end{array}\right\} \tag{27}$$

where the interval of integration is (a, b) and the initial conditions $y''(a)$ and $z''(a)$

$$y(\alpha, \mu; b) = y(\alpha_0, \mu_0; b) + \delta\alpha_0 \frac{\partial}{\partial\alpha} y(\alpha_0, \mu_0; b)$$
$$+ \delta\mu_0 \frac{\partial}{\partial\mu} y(\alpha_0, \mu_0; b) \tag{28}$$

$$z(\alpha_0, \mu; b) = z(\alpha_0, \mu_0; b) + \frac{\partial}{\partial\alpha} z(\alpha_0, \mu_0; b)$$
$$\delta\mu_0 \frac{\partial}{\partial\mu} z(\alpha_0, \mu_0; b) \tag{29}$$

where $\delta a_0 = a - a_0$ and $\delta\mu_0 = \mu - \mu_0$ are corrections for a_0 and μ_0 because a_0, μ_0 are only approximate values of a and μ and are not the exact values. In Eqns. (28) and (29) the left hand side is known because these are the prescribed values by y and z at $x = b$. The first terms on the right hand side in both the equations are also known because they may be obtained by numerical integration. To solve the equations for δa_0 and $\delta\mu_0$ we must know the partial derivatives contained in it. Since y and z are unknown analytical functions, the partial derivatives occurred in Eqns. (28) and (29) cannot be obtained analytically. These partial derivatives can be numerically found as follows.

If Δa_0 and $\Delta\mu_0$ are very small increments to a_0 and μ_0, respectively, then we can write.

$$\frac{\partial}{\partial\alpha}y(\alpha_0, \mu_0; b) = [y(\alpha_0 + \Delta\alpha_0, \mu_0; b) - y(\alpha_0, \mu_0; b)]/\Delta\alpha_0 \tag{30}$$

and

$$\frac{\partial}{\partial\mu}y(\alpha_0, \mu_0; b) = [y(\alpha_0, \mu_0 + \Delta\mu_0; b) - y(\alpha_0, \mu_0; b)]/\Delta\mu_0]$$

and two other similar expressions for $\delta/\delta a\ z(a_0, \mu_0; b)$ and $\delta/\partial\mu\ z(a_0\ \mu_0; b)$.

Since $y\ (a_0 + \Delta a_0, \mu_0; b)$, $Z\ (a_0 + \Delta a_0, \mu_0; b)$, $y\ (a_0, \mu_0 + \Delta\mu_0, b)$ and $Z\ (a_0, \mu_0 + \Delta\mu_0; b)$ can be evaluated numerically with the aid of Eqn. (30), all the partial derivatives appeared in Eqns. (28) and (29) become known. So the Eqns. (28) and (29) can be solved for $\delta\ a_0$ and $\delta\ \mu_0$ after replacing the left hand side by their actual prescribed values y_b and z_b. Thus the next better approximations a_1 and μ_1 can be obtained from $a_1 = a_0 + \delta a_0$ and $\mu_1 = \mu_0 + \delta\mu_0$. The process is repeated with a_1 and μ_1 to obtain a_2 and μ_2 which are more refined values of the missing initial conditions. This process is repeated until for some a_k and μ_k, $y(a_k, \mu_k; b)$ and $z(a_k, \mu_k; b)$ are coincided with y_b and z_b respectively to some desired degree of accuracy. Thus the process is stopped when $|\ y_b - y(a_k, \mu_k; b)\ | < \varepsilon$ and $|\ z_b - z(a_k, \mu_k; b)\ | < \varepsilon$ for some k. Here ε is taken as the measure of the accuracy. Then a_k and μ_k may be taken as the actual values of $y''(a)$ and $z''(a)$ respectively.

The convergence of the iteration process described above is not guaranteed and for every iteration the system of Eqns. (25) and (26) are to be integrated thrice. So good initial approximations for a and μ should be available.

Based on the above procedure an algorithm has been developed to solve the general third order boundary value problems consisting of a pair of simultaneous ordinary differential equations.

3.1 Algorithm

Step 1 : Let a_0 and μ_0 be two guessed values for the missing initial conditions and Δa_0 and $\Delta\mu_0$ are small increments to them.

Step 2 : Integrate the systems

$$y''' = f(x, y, y', y'', z, z', z'')$$

$$z''' = g(x, y, y', y'', z, z', z'')$$

with the prescribed initial conditions and

$$y''(a) = \alpha_k,\ z''(a) = \mu_k \text{ from } x = a \text{ to } x = b$$

$$(\ k = 0, 1, 2, \ldots\ )$$

Step 3 : (Call $y\ (a_k, \mu_k; b)$ and $z\ (a_k, \mu_k; b)$ the values of $y(b)$ and $z(b)$ respectively obtained on integration of the system in step 2.

Step 4 : Solve the system in step 2 with all the initial conditions being same except for $y''(a)$ and taking $y''(a) = a_k + \Delta a_k$. Let $y_{\Delta a_k}$ and $z_{\Delta a_k}$ be the values of y and z respectively at $x = b$ obtained on integration.

Step 5 Solve the system given in step 2 with all the initial conditions given there except $z''(a)$ and taking $z''(a) = \mu_k + \Delta\mu_k$. Let $y_{\Delta\mu_k}$, and $z_{\Delta\mu_k}$ are the values of y and z respectively at $x = b$ obtained on integration.

Step 6 Compute the partial derivatives

$$DY\alpha = [y_{\Delta\alpha_k} - y(\alpha_k, \mu_k; b)]/\Delta\alpha_k$$

$$DY\mu = [y_{\Delta\mu_k} - y(\alpha_k, \mu_k; b)]/\Delta\mu_k$$

$$DZ\alpha = [z_{\Delta\alpha_k} - z(\alpha_k, \mu_k; b)]/\Delta\alpha_k$$

$$DZ\mu = [z_{\Delta\mu_k} - z(\alpha_k, \mu_k; b)]/\Delta\mu_k$$

Step 7 Compute the corrections

$$\delta\alpha_k = [Dy\mu\cdot\{z(\alpha_k, \mu_k; b) - z_b\} \\ -Dz\mu\cdot\{y(\alpha_k, \mu_k; b) - y_b\}]/D$$

$$\partial\mu_k = [Dz\alpha\cdot\{y(\alpha_k, \mu_k; b) - z_b\} \\ -Dy\alpha\cdot\{z(\alpha_k, \mu_k; b) - y_b\}]/D$$

where

$$D = Dy\alpha\cdot Dz\mu - Dy\mu\cdot Dz\alpha$$

Step 8 Obtain the next approximation from

$$\alpha_{k+1} = \alpha_k + \delta\alpha_k;\ \mu_{k+1} = \mu_k + \delta\mu_k$$
$$(k = 0, 1, 2, \ldots)$$

Step 9 If

$$y_b - y(\alpha_k, \mu_k; b) \mid < \epsilon \quad \text{and}$$
$$z_b - z(\alpha_k, \mu_k, b) \mid < \epsilon$$

for a prescribed ϵ then go to step 2. Otherwise continue.

Step 10 : Solve the system of Eqns. (25) and (26) taking a_k and μ_k as the actual values of $y''(a)$ and $z''(a)$ respectively, treating it as an initial value problem.

In actual practice a sub-routine for the differential equations should be used. The guessed values for the missing initial conditions can be obtained from the physical considerations of the problem, from the prescribed boundary conditions or by a trial and error method. In the present problem, the results obtained by Hazarika[7] were taken. Though the convergence of the iteration process described earlier is not guaranteed but in actual practice it is experienced that if the guessed values are nearer to the true values, the convergence is rapid.

Since one of the boundaries of the present problem is at infinity, the problem becomes a singular one. The boundary at infinity was tackled in the following manner. For a particular pair of the values of the parameters a large value ξ_a of ξ is chosen and with the help of the iteration process described above the missing initial values are estimated. These values are, say, a_i and μ_i. Then the value of the ξ_a is increased and the a_i and μ_i are again estimated. If the latter values of a_i and μ_i do not differ significantly from the previous ones, then any of the ξ_a can be taken as the truncated boundary at infinity. If the latter values differ significantly from the previous values, then the process is repeated with increasing ξ_a and estimating a_i and μ_i each time until the consecutive pair of a_i and μ_i are identical at some significant places. Then the last values of ξ_a are taken to be the truncated boundary at infinity.

With the necessary modifications, the algorithm presented above is used to solve the boundary value problem given at Eqns. (21) to (24). The convergence also seems to be quite rapid.

4. RESULTS AND DISCUSSION

With the help of the algorithm, the boundary value problem is solved numerically for various values of the parameters β, S and λ.

Tables 1 (a) to 1 (c) represent the estimation of $F''(0)$ and $G''(0)$ for $\beta = -0.19884$, 0,1; $S = 0.1$, 0.5, 0.4 and $\lambda = 0.1$, 0.5, 0.4. These tables give the rapidity of the convergence of the iteration process. From these tables it is clear that the true values of the missing initial values are obtained at the maximum of fifth iteration (the first iteration is the guessed approximation).

Table 1 (a). Estimation of F'' and G'' at $\beta = -0.19884$, $S = 0.1$ and $\lambda = 0.1$

No. of iterations	$a_k = F''(0)$	$\mu_k = G''(0)$	$F'(\infty)$	$G'(\infty)$
1.	0.00846	0.009	0.885293	0.0638862
2.	0.168397	0.148388	1.0045	0.593908
3.	0.127065	0.232103	1.01324	1.07218
4.	0.116581	0.210372	1.00002	0.998387
5.	0.116328	0.210518	1.0	1.0
6.	0.116328	0.210518	1.0	1.0

Table 1 (b). Estimation of F' and G' at $\beta = 0$, $S = 0.5$ and $\lambda = 0.5$

k	a_k	μ_k	$F'(\infty)$	$G'(\infty)$
1.	0.378714	0.333405	1.04702	1.04938
	0.355886	0.31317	0.999476	0.999429
3.	0.356124	0.313392	1.0	1.0
4.	0.356124	0.313392	1.0	1.0

Table 1 (c). Estimation of F' and G' at $\beta = 1$, $S = 0.4$ and $\lambda = 0.4$

k	α_k	μ_k	$F'(\infty)$	$G'(\infty)$
1	1.2326	1.0	3.96004	4.03077
2.	0.979099	0.650724	1.3234	1.41408
3.	0.925909	0.536565	0.990561	1.01008
4.	0.924081	0.526044	0.999727	0.999612
	0.924135	0.526185	1.0	1.0

Tables 2(a) to 2(c) give the final solution of the problem for
$\beta = -0.19884, 0, 1;$
$S = 0.1, 0.4, 0.5;$ and
$\lambda = 0.2, 0.4, 0.5$
for $\beta = -0.19884, 0, 1$; $S = 0.1, 0.4, 0.5$, and $\lambda = 0.2, 0.4, 0.5$

We have computed the values of the functions F, F', G and G' with smaller size width (interval), but to reduce the size of the Tables we have presented the results here in the intervals of 0.4.

It is observed that both viscous boundary layer and magnetic boundary layer decrease as β increases. For a particular values of ξ, the velocity as well as magnetic field increase with increase of β. Thus the shooting method is preferable to obtain numerical solution where other methods seems to be laborious in mathematical treatment.

Table 2 (a). Estimation of F, F', G and G' at $\beta = -0.19884$, $S = 0.1$ and $\lambda = 0.2$.

ξ	F	F'	G	G'
0	0	0	0	0
0.4	0.0112103	0.0608088	0.0177159	0.088602
0.8	0.0524045	0.149738	0.0709964	0.178189
1.2	0.124462	0.264507	0.160966	0.273165
1.6	0.266713	0.399387	0.291336	0.381612
2.0	0.455312	0.544184	0.468817	0.508795
2.4	0.701537	0.700533	0.699994	0.657682
2.8	1.00088	0.827586	0.985982	0.789309
3.2	1.34376	0.921589	1.31995	0.905271
3.6	1.71794	0.984242	1.68969	0.977817
4.0	2.1156	0.99972	2.08213	0.99983
4.4	2.51298	0.999999	2.48319	0.999999

Table 2 (b). Estimation of F, F', G and G' at $\beta = 0$, $S = 0.4$ and $\lambda = 0.4$.

ξ	F	F'	G	G'
0	0	0	0	0
0.4	0.030289	0.15138	0.0266724	0.133362
0.8	0.120921	0.301299	0.106687	0.266694
1.2	0.270666	0.446186	0.239978	0.399603
1.6	0.476515	0.5809	0.426078	0.530231
2.0	0.733314	0.700113	0.663258	0.653995
2.4	1.03399	0.799803	0.947451	0.76412
2.8	1.3703	0.888154	1.27191	0.864527
3.2	1.73374	0.955725	1.62808	0.933651
3.6	2.11645	0.986076	2.00722	0.980854
4.0	2.51187	0.999982	2.40169	0.999807
4.4	2.92086	1.0	2.80303	1.0

Table 2 (c). Estimation of F, F', G and G' at $\beta = 1$, $S = 0.5$ and $\lambda = 0.5$.

ξ	F	F'	G	G'
0.4	0.061198	0.292958	0.0376098	0.187996
0.8	0.224569	0.512611	0.15014	0.373811
1.2	0.463243	0.671915	0.335242	0.548774
1.6	0.756005	0.785349	0.585827	0.608897
2.0	1.08705	0.865042	0.889596	0.813697
2.4	1.44472	0.929716	1.23196	0.902559
2.8	1.82034	0.966754	1.59979	0.953571
3.2	2.20754	0.988393	1.98337	0.983657
3.6	2.60186	0.999001	2.37625	0.998109
4.0	3.0004	0.999905	2.77446	0.999908
4.4	3.400156	1.0	3.17516	1.0

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