Def Sci J, Vol 41, No 4, October 1991, pp 373-383

Effect of Grievance on the Dynamics of Arms Race

R.K. Varma*

Institute for Systems Studies and Analyses, Delhi–110 054

ABSTRACT

An attempt has been made to study the effect of memory on the dynamics of arms race between two nations. This effect is represented by a function which accounts for the conflicting behaviour of each nation towards its adversary for the whole past upto the present. The solution of the resulting system of integro-differential equations is characterised by the presence of damped and undamped oscillations.

1. INTRODUCTION

Research on arms race is related to that aspect of social science which deals with the behaviour of human groups under various political, social and economic conditions. Because of the very complicated nature of these factors, studies in this area have been mostly observational and non-mathematical. The first attempt at a quantitative description of armament process was made by Richardson[1,2]. His mathematical model of the theory of war provided a great impetus to research activity in the areas of social science related to conflict and cooperation, international relations, etc resulting in the appearance of various research papers dealing with such topics[3,4].

The problem of arms race is intimately connected with the problem of war between nations and its consequent effects on economics, foreign policy and the group attitudes. Despite its complexities, in general, it may be considered as an interactive process between the military programmes of different nations. Thus it may be defined as an activity carried out by two or more potential adversaries to increase and improve their armaments in view of the past, present and future behaviours of the other party.

Received 26 June 1990, revised 10 January 1991

* *Present Address : JA-5/B, Ashok Vihar Phase I, New Delhi-110 052.*

However, it is not wholly a simple action and reaction process, although each side remains sensitive to the technological advances of the other and tries to do what fits best in accordance with its own political belief, strategy and national interest.

The gross behaviour of the enemy nations having continuing grievance may be explained to a great extent within the framework of Richardson model of arms race. A cumulative effect of their mutual perception, which may be termed as memory, for lack of a better terminology, may have a critical influence on their present attitudes. Much as it happens to an individual, in the case of nations, a part of this experience is obliterated with time and the remaining part having serious military or political import retained which might have a potential influence in moulding the present and near future international relations. In the following study an attempt has been made to incorporate this historical perception in the dynamics of arms race.

2. FUNDAMENTAL EQUATIONS

The differential equations of arms race between two nations X and Y, incorporating the effect of past history on their present dynamics, may be described by the following :

$$\dot{x} = ky - \alpha x + k_1 g$$

$$\dot{y} = lx - \beta y + l_1 h \tag{1}$$

where

$$g = a \int_I^t e^{-a(t-\tau)}\, y(\tau)\, d\tau$$

$$h = b \int_I^t e^{-b(t-\tau)}\, x(\tau)\, d\tau$$

with the initial conditions

$$t = 0, \quad x = x(0), \quad y = y(0), \quad g = g(0), \quad h = h(0) \tag{2}$$

The variables x and y may be interpreted as defence expenditure or the level of armaments of the nations X and Y at time t. The function $g(t)$ represents the integral of nation Y's conflictual behaviour during the interval $(-T, t)$, i.e., a past extending upto $-T$ to the present time t. Its equivalent differential form is

$$g(t) = a[y(t) - g(t)] \tag{3}$$

The parameter a known as the forgetting factor of Y refers to the speed of adjustment of the past to the present.

A first order analysis of Eqn (3) for large and small values of a, results in the following :

$$g(t) \simeq y(t), \quad a >> 0$$

$$g(t) \simeq \text{a constant}, \quad a \simeq 0 \tag{4}$$

This shows that, for $a >> 0$ the speed of adjustment is very fast and consequently remote past has little influence whereas for $a \simeq 0$ it is slow and therefore, recent past has more influence.

Similarly, the function $h(t)$ may be interpreted in a differential form

$$\dot{h}(t) = b(x - h) \tag{5}$$

The other parameters, viz. k, l are termed as defence coefficients; α, β fatigue or expense coefficients; and k_1, l_1 grievance coefficients. The units of all these parameters is s^{-1}. For mathematical simplicity in analyses they are assumed here as constants.

For further analysis, considering Eqns (1)–(3) and (5), and using the following variables,

$$t' = t/t_0; \quad x' = x/x(0); \quad y' = y/x(0), \quad g' = g/x(0); \quad h' = h/x(0) \tag{6}$$

to obtain their non-dimensional forms given below :

$$\begin{aligned} \dot{x}' &= -\alpha' x' + k' y' + k_1' g' \\ \dot{y}' &= l' x' - \beta' y' + l_1' h' \\ \dot{g}' &= a'(y' - g') \\ \dot{h}' &= b'(x' - h') \end{aligned} \tag{7}$$

The initial conditions are

$$\begin{aligned} &t' = 0; \quad x' = 1; \quad y' = y(0)/x(0) = Y(0); \\ &g' = g(0)/x(0) = g'(0); \quad h' = h(0)/x(0) = h'(0) \end{aligned} \tag{8}$$

where

$$g'(0) = a' \int_{T'}^{0} e^{a'\tau'} y'(\tau')d\tau'; \quad h'(0) = b' \int_{-T'}^{0} e^{b'\tau'} x'(\tau')d\tau'$$

$$\begin{aligned} &\alpha' = \alpha t_0; \quad \beta' t_0; \quad k' = k t_0; \quad l' = l t_0; \quad k_1' = k_1 t_0; \\ &l_1' = l_1 t_0; \quad a' = a t_0; \quad b' = b t_0; \quad \tau' = \tau t_0; \quad T' = T t_0 \end{aligned} \tag{9}$$

(For clarity the notation of prime upon the variables is omitted hereafter)

It may be observed that Eqn (7) represent a coupled system of linear differential equations which may be reduced to a single higher order equation in terms of any one of the dependent variables in Eqn (7) for obtaining their solution. But, sometimes, it is more convenient to solve a few lower order equations than a single higher order one[5]. In view of this, the equations are retained as such.

Equation (7) can be represented in the matrix form as follows

$$\dot{Z} = AZ \tag{10}$$

with

$$Z_{t=0} = Z(0)$$

where

$$Z = [x, y, g, h]^T \tag{11}$$

$$A = \begin{bmatrix} -\alpha & k & k & 0 \\ l & -\beta & 0 & l \\ 0 & a & -a & 0 \\ b & 0 & 0 & -b \end{bmatrix} \tag{12}$$

and

$$Z(0) = [1, Y(0), g(0), h(0)]^T \tag{13}$$

3. SOLUTION OF EQUATIONS

Assuming[6]

$$Z = e^{\lambda t} v \tag{14}$$

and substituting in Eqn (10), we get

$$(A - \lambda I)v = 0 \tag{15}$$

where λ and v are the eigenvalues and eigenvector of A and I is the unit matrix of the same order as A.

For a non-trivial solution of Eqn (15), i.e., $v \neq 0$

$$A - I \mid = 0 \tag{16}$$

Equation (16) is a 4th degree polynomial in λ of the form

$$\lambda^4 + (a + b + \alpha + \beta)\lambda^3 + [ab + (a + b)(\alpha + \beta) + \alpha\beta - lk]\lambda^2 + [(a + b)(\alpha\beta - kl)$$

$$+ ab(\alpha + \beta) - (k_1 la + kl_1 b)\lambda + ab[\alpha\beta - (k + k_1)(l + l_1)] = 0 \tag{17}$$

It has four roots λ_i $(i=1,2,3,4)$ which may be multiple, real and complex conjugates. Corresponding to each of these roots, Eqn (15) is solved to obtain the eigen/vectors v, with components $v_i (i=1,2,3,4)$ leading to the solution of Eqn (10) as

$$Z = \sum_{i=1}^{4} C_i e^{\lambda_i t} v_i \tag{18}$$

The eigenvectors v_i corresponding to each λ_i are given by the solution of the matrix Eqn (15), i.e., $(A - \lambda_i I)v'_i = 0$. This leads to $v_i = (v^i_{11}, v^i_{12}, v^i_{13}, v^i_{14})$, as a (4*1) column vector where v_{11}^i, v_{12}^i, etc are the co-factors corresponding to the elements (say) of the first row of the determinant shown in Eqn (16) calculated for each λ_i. C_i are the constants of integration which are determined from the initial conditions [Eqn (13)]. The solution to Eqn (18) for some typical values of the parameters is given in Section 5.

4. STABILITY OF EQUILIBRIUM SOLUTION

For a dynamic system determined by a set of differential equations, it is of practical interest to know whether its solution is stable or not. This question is, generally, resolved by determining whether there exists equilibrium solution Ze, and if so, whether it is stable or unstable. Further, whether $Z(t)$ approaches Ze for large t. The solution Ze constitutes an important feature of the dynamical equations. It represents the set of values of the state variables for which the rates of change of states are simultaneously zero. If a system ever attains such values, it will no longer evolve unless disturbed externally. For the set of Eqns (10), Ze is thus given by solving the equations :

$$AZ = 0 \tag{19}$$

If the determinant $|A| \neq 0$, Eqn (19) leads to the trivial solution

$$Z = Ze = [x, y, g, h]^T = 0 \tag{20}$$

On the other hand if $|A| = 0$, Eqn (19) has an infinite number of non-trivial solutions

$$x = h = \frac{(k + k_i)C}{\alpha} = \frac{\beta C}{l + l_1}; \quad y = g = c \tag{21}$$

where C is arbitrary.

The stability of Eqns (20) and (21) is determined by the roots $\lambda_i (i = 1,2,3,4)$ of the characteristic polynomial Eqn (17). For specific values of the parameters, it is possible to find the nature of the roots and hence determine the stability or otherwise of the equilibrium solution. For general values of the parameters, Routh-Hurwitz criteria for stability[7] may be used. According to this, for stable solution the polynomial (Eqn (17)) has all roots with negative real parts, provided all the principal minors of the Routh-Hurwitz matrix :

$$H = \begin{vmatrix} a_1 & & 0 & 0 \\ a_3 & a_2 & a_1 & \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{vmatrix} \tag{22}$$

The elements indicated by the dotted lines are positive.

Here a_1, a_2, a_3 and a_4 are the coefficients of λ^3, λ^2, λ and λ^0, i.e., the constant term in the Eqn (17).

It is obvious that the above procedure provides the combined effect of the general values of the parameters on the stability of the solution from which it is not possible to estimate the effect of each parameter separately. This can however, be achieved through a simple analytical solution of Eqn (7) which may be obtained for the special case when

$$\alpha = \beta, \quad k = l, \quad k_1 = l_1, \quad a = b \tag{23}$$

Thus, using Eqn (23) in Eqn (7) and solving the resulting differential equations we get

$$2x = A_1(e^{S_1 t} - e^{S_2 t}) + A_2(e^{\gamma_1 t} - e^{\gamma_2 t}) + p(0)\, e^{S_2 t} + p_1(0)\, e^{\gamma_2 t}$$

$$2g = B_1(e^{S_1 t} - e^{S_2 t}) + B_2(e^{\gamma_1 t} - e^{\gamma_2 t}) + q(0)\, e^{S_2 t} + q_1(0)\, e^{\gamma_2 t} \tag{24}$$

where the constants

$p(0), p_1(0), q(0), q_1(0), S_1, S_2, \gamma_1, \gamma_2, A_1, A_2, B_1, B_2$ are as follows

$$p(0) = \ + Y(0); \quad p_1(0) = \ - Y(0)$$

$$q(0) = g(0) + h(0), \quad q_1(0) = g(0) - h(0)$$

$$S_1, S_2 = [-(a + \alpha - k) \pm \sqrt{D}]/2$$

$$\gamma_1, \gamma_2 = [-(a + \alpha + k) \pm \sqrt{D_1}]/2$$

$$A_1 = (k_1 q(0) + p(0)\,(a + S_1))/\sqrt{D}; \quad A_2 = (k_1 q_1(0) + p_1(0)\,(a + \gamma_1))/\sqrt{D_1}$$

$$B_1 = (ap(0) + q(0)\,(S_1 + \alpha - k)/\sqrt{D}; \quad B_2 = \ ap_1(0) + q_1(0)\,(\gamma_1 + \alpha + k))/\sqrt{D_1}$$

$$D = (a + \alpha - k)^2 - 4a(\alpha - k - k_1); \quad D_1 = (a + \alpha + k)^2 - 4a(\alpha + k + k_1) \tag{25}$$

Similar expressions for y and h follows. From the Eqns (24) and (25), it may be observed that

(i) When $a > 0$

- (a) both S_1 and S_2 are real and unequal
- (b) for $a + (k > k_1)$, $(S_1, S_2) < 0$; and for $a = k + k_1$, $S_1 = 0$, $S_2 < 0$
- (c) for $a < (k + k_1)$, $S_1 > 0$, $S_2 < 0$
- (d) for $D_1 \geq 0$, both γ_1 and γ_2 are negative
- (e) when $D_1 < 0$, both γ_1 and γ_2 are complex conjugates

In view of (b), (d) and (e); x, $g \to 0$ as $t \to \infty$. Hence the solution is stable. However, the solution is no longer stable in case (c) as x, $g \to \infty$ as $t \to \infty$.

(ii) When $a = 0$; one gets

$$S_1 = \gamma_1 = 0, \quad S_2 = -(\alpha - k), \quad \gamma_2 = -(\alpha + k)$$

Here $x \to 0$ as $t \to \infty$ when $a \geq k$ and $x \to \infty$ as $t \to \infty$ when $a < k$ and g reduces to constant $g(0)$, its initial value for $a \geq k$.

Thus in the presence of grievance characterised by the parameter 'a', the solutions are combinations of overdamped, and underdamped oscillations as well as undamped ones leading to their being stable or unstable as the case may be. But in the absence of 'a', these solutions are either overdamped or undamped with no oscillation.

5. EXAMPLE

Let $a = \beta = 2$, $a = 2$, $b = 3$, $k = l = k_1 = l_1 =$ Substituting these values in Eqn (17) we get

$$\lambda(\lambda^3 + 9\lambda^2 + 29\lambda + 34) = 0$$

which by solving gives

$$\lambda_1 = 0, \quad \lambda_2 = -3.4534, \quad \lambda_3 = X_1 + iY_1, \quad \lambda_4 = X_1 \quad iY_1$$

where

$$X_1 = -2.7734, \quad Y_1 = 1.4678, \quad i = \sqrt{-1} \tag{27}$$

Evaluating $g(0)$, $h(0)$ and the constants C using Eqn (18), the following solutions, are obtained :

$$x(t) = 0.9685 + 0.5540e^{-2.7734t} \cos [84.0987t - 273.27]$$

$$y(t) = 0.9685 + 0.7970e^{-2.7734t} \cos [84.0987t - 87.8127]$$

$$g(t) = 0.9685 + 0.9607e^{-2.7734t} \cos [84.0987t - 205.60]$$

$$h(t) = 0.9685 + 1.1191e^{-2.7734t} \cos [84.0987t - 354.4876] \tag{28}$$

for the initial conditions

$$x(0) = 1; \quad Y(0) = 1; \quad g(0) = 0.1029; \quad h(0) = 2.0832 \tag{29}$$

and

$$C_1 = -0.9685; \quad C_2 = 0; \quad C_3 = 0.0870; \quad C_4 = 0.1039$$

and

$$x(t) = 07755 + 0.4255e^{-2.7734t} \cos (84.0987t - 302.2409)$$

$$y(t) = 0.7755 + 0.1622e^{-2.7734t} \cos (84.0987t - 116.7890)$$

$$g(t) = 0.7755 + 0.7380e^{-2.7734t} \cos (84.0987t - 234.5744)$$

$$h(t) = 0.7755 + 0.8596e^{-2.7734t} \cos (84.0987t - 23.4651) \tag{30}$$

For the initial conditions

$$x(0)=1; \quad Y(0)=0.5; \quad g(0)=0.3453; \quad h(0)=1.5649 \tag{31}$$

and

$$C_1=-0.7755; \quad C_2=0; \quad C_3=0.0198; \quad C_4=-0.1022$$

with arguments of cosine in degrees.

6. SOLUTION FOR CONSTANTS g AND h

For comparison with the results of previous section where $g(t)$ and $h(t)$ are assumed variables, here the solution is presented when g and h are constants throughout. The differential equations for this case are given by the first two equations of Eqn (7) along with the initial conditions

$$x(0)=1; \quad y(0)=Y(0)$$

The solutions are as follows :

$$x(t)=(\beta k_1 g+kl_1 h)/(S_1 S_2)+[\{S_1+\beta+k_1 g+kY(0)+(\beta k_1 g+kl_1 h)/S_1\}\ e^{S_1 t}$$

$$-\{S_2+\beta+k_1 g+kY(0)+(\beta k_1 g+kl_1 h)/S_2\}\ e^{S_2 t}]/(S_1-S_2)$$

$$y(t)=(\alpha l_1 h+k_1 lg)/S_1 S_2+[\{(S_1+\alpha)Y(0)+l_1 h+l+(\alpha l_1 h+k_1 lg)/S_1\}\ e^{S_1 t}$$

$$-\{(S_2+\alpha)Y(0)+l_1 h+l+(\alpha l_1 h+k_1 lg)/S_2)\ e^{S_2 t}]/(S_1-S_2) \tag{32}$$

where the roots S_1 and S_2 of the equation are

$$S_2+S(\alpha+\beta)+(\alpha\beta-kl)=0 \tag{33}$$

$$S_1, S_2=-[(\alpha+\beta)\pm\sqrt{(\alpha+\beta)^2-4(\alpha\beta-kl)}]/2 \tag{34}$$

The corresponding result for zero grievances ($g=h=0$) can be deduced from Eqn (32).

It may be noted that both the roots S_1 and S_2 are real and not equal to zero. Further, $S_1 \leqslant 0$ and $S_2<0$, when $\alpha\beta \geqslant kl$ which leads to both x, $y \to 0$ as $t\to\infty$. However, $S_1>0, S_2<0$ when $\alpha\beta<kl$ making x, y increase indefinitely with t.

To compare the results with the variables g and h given earlier, $x(t)$ and $y(t)$ are evaluated for the same set of initial conditions given in Eqns (29) and (31). These are

$$x(t)=0.7630-0.0931e^{-t}+0.3301e^{-3t}$$

$$y(t)=1.4231-0.0931e^{-t}-0.3301e^{-3t} \tag{35}$$

corresponding to Eqn (29) and

$$x(t)=0.7518-0.2051e^{-t}+0.4533e^{-3t}$$

$$y(t)=1.1584-0.2051e^{-t}-0.4533e^{-3t} \tag{36}$$

corresponding to Eqn (31).

The solution curves for these cases are given in Figs. 1 and 2 respectively

7. CONC USION

It may be noted (Figs. 1 and 2) that x and y, as also g and h stabilise asymptotically to the same value and continue to stay at that level. These levelling up follows the pattern of damped oscillation with a period $2/Y_1$, (Eqn (27)) as their analytical solutions indicate. Further, in the initial stages there is a fall in the values of x, though small, despite the presence of $g(t)$, the grievance due to Y. However as soon as $g(t)$ increases beyond a certain level, $x(t)$ shows an increase till both x and g asymptotically become the same. During the same period y increases because of greater hostility $h(t)$ of X towards Y. However, soon $y(t)$ starts decreasing till it asymptotically comes to the same level as $x(t)$. Finally beyond $t \geqslant 1.6$, the variables x,y,g and h asymplotically tend to the same value.

For the case of constant g and h, and the stable case $a\beta \geqslant kl$, the variables $x(t)$, $y(t)$ show an initial decrease and increase respectively and ultimately level off asymptotically to different values, which is unlike the case when g and h are assumed variables.

In view of the stability considerations given in sections 4 and 6, it is inferred that in the presence of variable grievances $g(t)$ and $h(t)$ the solutions $x(t)$ and $y(t)$ are

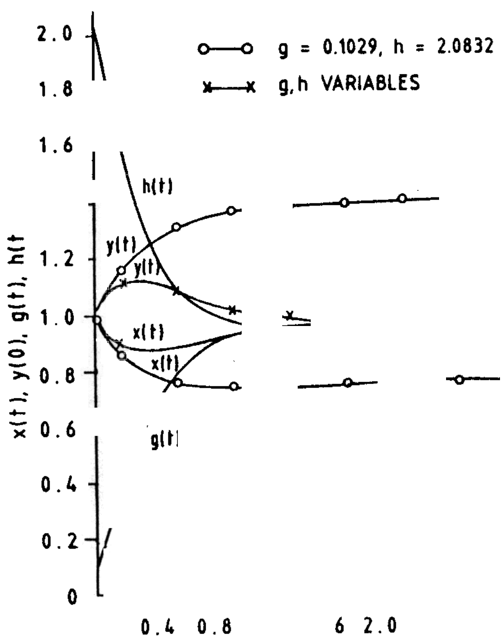

Figure 1. Variations in arms levels $x(t)$, $y(t)$ of two nations X and Y and their respective grievances $g(t)$ and $h(t)$ with time t, when $x(0) = y(0) = 1$, $g(0) = 0.1029$ and $h(0) = 2.0832$.

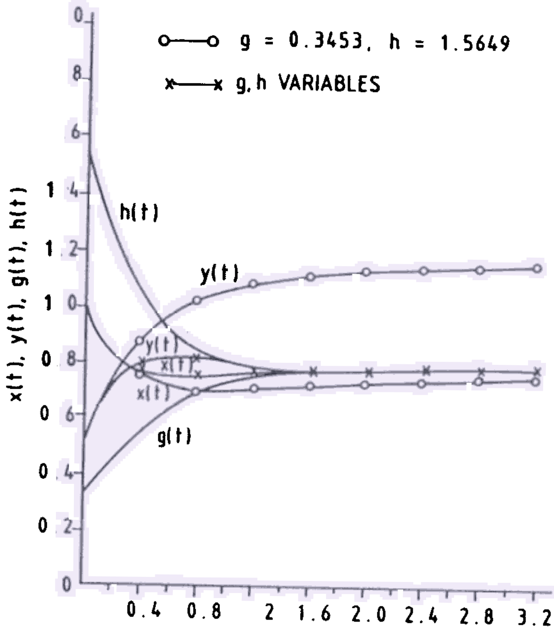

Figure 2. Variations in arms levels $x(t)$, $y(t)$ of two nations X and Y and their respective grievances $g(t)$ and $h(t)$ with time t, when $x(0) = 1$, $y(0) = 0.5$, $g(0) = 0.3453$ and $h(0) = 1.5649$.

oscillatory, damped as well as undamped in character whereas for constant g and h, including zero grievances, these are undamped or damped. However, in both the cases there are situations where $x(t)$ and $y(t)$ are unstable as well as stable leading in the former case to a runway arms race with $x(t)$ and $y(t)$ increasing indefinitely with time, and in the latter case, stabilise to their equilibrium values with no further accumulation of arms.

ACKNOWLEDGEMENTS

The author is thankful to Dr NK Jaiswal, Director, Institute for Systems Studies and Analyses for his keen interest and useful suggestions during the formulation and solution of the problem. Thanks are also due to Mrs Kamala Martolia for her assistance in computation.

REFERENCES

1. Richardson, L.F. Generalised foreign politics. *British J. of Psychology*, 1939, monograph supplement no. 23.
2. Richardson, L.F. Arms and insecurity: a mathematical study of the causes and origins of war, edited by N. Rashevsky and E. Trucco. The Boxwood Press, Pittsburgh, 1960.
3. Isard, W. Arms races, arms control and conflict analysis. Cambridge University Press, New York, 1988. pp. 48-49.

4 Schmidt, C. (Ed). The economics of military expenditure. Macmillan Press, New York, 1987.

5. Pearson, C.E. (Ed). Handbook of applied mathematics. Van Nostrand Reinhold Company, New York, 1974. p. 314.

6. Huntley, I.D. & Johnson, R.M. Linear and nonlinear differential equations. John Wiley & Sons, New York, 1984. pp. 22-55.

7. Leipholz, H. Stability theory: an introduction to the stability of dynamic systems and rigid bodies. Academic Press, New York, 1970.