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Renewal Theory Applications to Continuous Inspection of Markovian Production Processes

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ABSTRACT

Theory of renewal processes has been used to study the effectiveness of three continuous sampling schemes, when the quality of the successive units in a continuous production process follows a two-state time-homogeneous Markov chain model which comprises the iid Bernoulli model considered by Dodge. The average outgoing quality and operating characteristics functions have been formulated and some numerical results have been given when the serial correlation coefficient of the Markov chain is assumed to be known *a priori*.

1. THE MODEL AND ASSUMPTIONS

The produced units are indexed by n. Let $X_n = 0$ or 1 depending on whether the nth unit produced is conforming (nondefective) or otherwise.

Assumption 1 : $\{X_n, n \geqq 0\}$ follows a two-state time-homogeneous Markov chain (MC) with transition probabilities.

$$P_{00} = 1 - a, P_{01} = a \quad (1)$$

$$P_{10} = \beta, P_{11} = 1 - \beta \quad (2)$$

Assumption 2 : The zeroth unit is assumed to be nonconforming and $P[X_0 = 1] = 1$.

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Assumption 3: The inspected unit that is found to be nonconforming is replaced by conforming unit.

Let $a + \beta = \delta$. Then $p = a\,\delta^{-1}$ is the long run proportion of nonconforming units. In fact, (p,q) (where $q = \beta\delta^{-1}$) is the stationary distribution for the transition probabilities in Eqn (1). The permanent $\phi = (1 - \delta)$ is the serial correlation coefficient between X_n and X_{n+1} ($n \geqq 0$) provided the stationary distribution is taken as the initial distribution.

With the assumption that $P[X_0 = 1] = 1$, together with the strong Markov property of the MC essentially implies that completion of an implementation of a continuous sampling plan (CSP) is a recurrent event. That is, the point at which $P[X_0 = 1] = 1$ is a regenerative point where renewal takes place. Observe that a renewal process is regenerative.

We make it a convention that, the zeroth unit is not counted in the computation of average outgoing quality (AOQ) and operating characteristic (OC).

2. FORMULATION

A CSP (CSP-1 or CSP-2 or MLP-2) is imposed on the production line. The CSP starts at item $X_0 = 1$ with full inspection until a success run of length r of coforming units are observed[1] and then the manufacturer switches to fractional sampling. Let T_1 be the number of units produced during the first full inspection period. We have

$$T_1 = \min\{n \geqq r : X_{n-r+1} = \dots\dots\dots = X_n = 0\} \tag{3}$$

Similarly, let M_1 be the number of units produced during the subsequent fractional sampling. The stopping time under fractional sampling varies from one sampling plan to another.

Procedures of CSP-1, CSP-2 and MLP-2 have been described in Dodge[2], Dodge and Torrey[3], and Lieberman and Solomon[4]. CSPs are used when the production is continuous and the formation of inspection lots for lot-by-lot inspection is artificial or impractical as in manufacturing industries like (i) ammunition loading and component manufacture, and (ii) confectionery and food industries.

The objective of CSPs is to guarantee a limiting value of AOQ called average outgoing quality limit (AOQL). The concept of continuous sampling inspection and the mathematical basis for CSP-1 were first presented by Dodge[2]. He studied the behaviour of CSP-1, CSP-2 and CSP-3 under the assumption of statistical control (i.e., the probability of finding a nonconforming unit is constant over the time axis). The procedure of CSP-1 is as follows:

(a) At the start, inspect 100 per cent of the units until r consecutive units are found to be conforming;

(b) When such a run of length r of conforming units are observed, discontinue 100 per cent inspection and inspect only a fraction of units selecting one unit at random from each block of k units; and

(c) When a nonconforming unit is observed under fractional sampling, revert immediately to 100 per cent inspection of succeeding units as per the above procedure and correct or replace all nonconforming units found.

The striking features of this plan are, (i) its heavy dependence on the occurrence of a single nonconforming unit which may be isolated, and (ii) the assumption of statistical control which is totally unrealistic.

The abrupt change between 100 per cent inspection and fractional sampling inspection may lead to difficulties in personnel assignments in the administration of the inspection process. For example, in a very complicated and expensive item such as an aircraft engine, this transition may require major readjustments.

Continuous sampling of the units produces renewal cycles (cycle is the period where full inspection begins to the epoch and is reverted again to full inspection). In each cycle we observe a pair of random variables (T_j, M_j) for $j = 1, 2.........$ Let $W_j = T_j + M_j$. Note that W_j is the number of units produced in the jth renewal cycle. It is also observed that, there is an unobservable random variable V_j which is associated with W_j; where V_j is the number of uninspected outgoing nonconforming units in the jth renewal cycle. Let t be the length of a production run and N_t is the number of renewal inspection cycles completed in the production run of length t: Then $\{N_t, t \geqq 0\}$ forms a discrete renewal process. Divide the discrete interval [0, t] into N_t renewal intervals and a possible incomplete $(N_t + 1)$th interval $[S_{N_t}, 1]$ where

$$S_{N_t} = \sum_{j=1}^{N_t} W_j$$

Let V_t be the number of uninspected outgoing nonconforming units in $[S_{N_t}, 1]$. V_t is also unobservable like V_j. It is necessary to distinguish a natural renewal interval and the last incomplete one, because of the different probability structures of the two.

The above formulation is based on Yang[5]. We now define :

$$AOQ(t) \quad = \quad E\left(\sum_{j=1}^{N_t} V_j + V_t\right) t^{-1} \text{ for } t = 1, 2, ... \tag{4}$$

By the strong Markov property of $\{X_n, n \geqq 0\}$, $\{V_j, j \geqq 1\}$, $\{T_j, j \geqq 1\}$, $\{M_j, j \geqq 1\}$, $\{W_j, j \geqq 1\}$, and $\{V_t, t \geqq 1\}$ are iid sequence. Hence by strong law of large numbers and by renewal theory we have

$$AOQ \quad = \limsup_{t \to \infty} AOQ(t) = E(V_1)/E(W_1) \tag{5}$$

We now define $OC(1)$ as the per cent of product units accepted without inspection.

Hence

$$OC(1) = p^{-1}\, E(V_1)/E(W_1) \tag{6}$$

Also, we define $OC(2)$ as the per cent of product units accepted on a sampling basis. Hence

$$OC(2) = E(M_1)/E(W_1) \tag{7}$$

It must be noted that, under Markovian assumption, the AOQ and other expressions of a particular CSP would depend on the type of fractional sampling procedure used (such as systematic sampling and probability sampling procedures). It should be pointed out that random sampling in CSPs for Markovian production processes seems absolutely intractable for any mathematical discussion.

Using systematic sampling procedure (it involves inspecting every kth unit from the flow of products in the production line), the expressions for $E(V_1)$, $E(W_1)$ and $E(M_1)$ are found (Table 1) (for derivations see Sampath Kumar and Rajarshi[6]).

Table 1.

Plan	$E(V_1)$	$E(W_1)$	$E(M_1)$
CSP-1	$(k\delta - G)/G$	$\frac{1-S}{p\delta S} + \frac{k}{pG}$	k/pG
CSP-2	$\frac{A_1[(D-E)(1+B)+1]+A_2(C+E)}{(1-E)}$	$\frac{1-S}{p\delta S} + \frac{k(1-E+B+D)}{B(1-E)}$	$\frac{k(1-E+B+D)}{B(1-E)}$
MLP-2	$\frac{A^r[A_3+A_2(C+J)-A_1(1+JB)]+A_1[1-JB^2]}{1-J}$	$\frac{1-S}{p\delta S} + (kF_1+k^2F_2)$	$kF_1+k^2F_2$

MLP = Multi level plan.

where

$$A = p_{00}^{(k)},\ B = 1 - A,\ C = p_{11}^{(k)},\ D = 1 - C \tag{8}$$

$$E = DA^{r-1},\quad A_1 = \sum_{h-1}^{k-1} p_{01}^{(h)}/p_{01}^{(k)},\quad A_2 = 1 \sum_{h-1}^{k-1} p_{11}^{(h)} \tag{9}$$

$$F_1 = [1 - A^r + BA^{r-1}(AC - DB)]/B \tag{10}$$

$$F_2 = A^r/E_1(1 - J),\ E_1 = p_{01}^{(k^2)},\ J = DA^{r-1} \tag{11}$$

$$A_3 = \sum_{h-1}^{k^2-1} p_{01}^{(h)}/p_{01}^{(k^2)},\ G = [1 - (1 - \delta)^k] \tag{12}$$

and

$$S = q\,(1 - p\delta)^{r-1} \tag{13}$$

3. NUMERICAL RESULTS

In this paper CSP-1, CSP-2 and MLP-2 were chosen for illustration. For a given r and k Table 2 compares the AOQL values for different value δ along with the

Table 2. Comparison of AOQL values in percentage for $k = 7$ when the serial correlation coefficient is known

Plans	δ								
	0.0001	0.0900	0.1500	0.2600	0.5400	0.6900	0.9100	UAOQL	r
CSP-1	0.007465	1.1184	1.1825	1.2209	1.1794	1.1231	1.0338	6.25	89
	0.007483	2.0417	2.2702	2.4226	2.3939	2.2892	2.1137	12.00	43
	0.007488	2.7066	3.1406	3.4533	3.4852	3.3457	3.0994	18.18	29
CSP-2 ($l = r$)	0.022359	1.6002	1.4411	1.3355	1.2198	1.1466	1.0430	8.88	121
	0.022446	3.9908	3.8161	3.6248	3.3544	3.1633	2.8836	21.05	43
	0.022450	4.2275	4.0710	3.8802	3.5962	3.3925	3.0937	22.22	40
MLP-2	0.071585	1.1225	1.0886	1.0813	1.0920	1.0696	1.0206	5.08	128
	0.073798	3.2053	3.1338	3.1284	3.1864	3.1314	2.9932	13.71	43
	0.074119	4.3425	4.2648	4.2688	4.3686	4.3000	4.1160	18.42	31

Plans	δ								
	0.9500	0.9750	1.0000	1.0250	1.0900	1.1300	1.1800	UAOQL	r
CSP-1	1.0176	1.0085	1.0000	0.95020	0.01115	0.00029	0.00000	6.25	89
	2.0826	2.0635	2.0389	2.02563	0.82318	0.18850	0.02100	12.00	43
	3.0548	3.0274	3.0000	2.97344	2.46196	1.23052	0.33977	18.18	29
CSP-2 ($l = r$)	1.0245	1.0146	1.0000	0.85908	0.00104	0.00000	0.00000	8.88	121
	2.8360	2.8070	2.7787	2.75073	1.49195	0.37059	0.04196	21.05	43
	3.0425	3.0116	2.9813	2.95141	1.87393	0.55601	0.07633	22.22	40
MLP-2	1.0112	1.0049	1.0000	0.56901	0.00027	0.00000	0.00000	5.08	128
	2.9656	2.9477	2.9300	2.91280	0.96000	0.19571	0.02117	13.71	43
	4.0784	4.0542	4.0308	2.65948	2.91971	1.10339	0.23938	18.42	31

unrestricted AOQL values of CSP-1, CSP-2 and MLP-2. We observe that for large values of r and small values of k (for example, $k = 5$ and $r = 15$), there is no significant difference in the AOQL values for small departures of δ from unity. At the same time, for small values of r and large values of k (for example, $r = 10$, $k = 11$), there is significant difference in the AOQL values for small departures of δ from unity. Hence, for large values of r and small values of k, one may conclude that CSP-1, CSP-2 and MLP-2 are robust; whereas for small values of r and large values of k, they need not be robust.

To compare the per cent of total production accepted on a sampling basis for 1 per cent AOQL and $k = 10$, a comparison of OC(2) values for $\delta = 0.50$, 1.00 and 1.10 is provided in Table 3. For $\delta < 1$, $k = 10$ and 1 per cent AOQL we find that for $p < p^*$ (the maximising value of p for which 1 per cent AOQL is attained), OC(2) is highest in MLP-2 and least in CSP-1; whereas for $p > p^*$, OC(2) is highest in CSP-1 and least in MLP-2 for the first few values of p starting from p^*. For $\delta > 1$ we find that OC(2) is highest in CSP-1 and least in MLP-2 (OC(2) is higher in CSP-2 than that in MLP-2 for the first few values of p, starting from p^*). Hence for $\delta < 1$ and $p < p^*$,

Table 3. Comparison of $OC(2)$ values for 1% AOQL and k = 10 when the serial correlation coefficient is known

	$\delta = 0.50$			$\delta = 1.00$			$\delta = 1.10$		
p	CSP-1 (r = 142)	CSP-2 (r = 196)	MLP-2 (r = 178)	CSP-1 (r = 109)	CSP-2 (r = 141)	MLP-2 (r = 152)	CSP-1 (r = 42)	CSP-2 (r = 49)	MLP-2 (r = 44)
0.00000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000			
0.00500	0.8187	0.8723	0.9860	0.9322	0.9657	0.9884			
0.01000	0.7351	0.7771	0.9120	0.8340	0.8781	0.9127			
0.01375	0.6655	0.6987	0.7985	0.7395	0.7758	0.7683*			
0.01500	0.6412	0.6707	0.7524	0.7046	0.7349	0.7030			
0.01625	0.6166	0.6418	0.7042*	0.6683	0.6912	0.6324			
0.01750	0.5918	0.6121	0.6554	0.6309	0.6450*	0.5597			
0.01875	0.5668	0.5817	0.6070	0.5927*	0.5971	0.4878			
0.02000	0.5417	0.5509	0.5600	0.5542	0.5481	0.4196			
0.02125	0.5167	0.5198*	0.5151	0.5156	0.4989	0.3569			
0.02375	0.4672*	0.4575	0.4327	0.4398	0.4032	0.2518			
0.03000	0.3507	0.3114	0.2724	0.2727	0.2074	0.0977			
0.04000	0.2024	0.1426	0.1221	0.1057	0.0553	0.0202			
0.05000	0.1075	0.0573	0.0517	0.0361	0.0129	0.0041			
0.07000	0.0266	0.0079	0.0084	0.0037	0.0006	0.0002			
0.09000	0.0061	0.0010	0.0013	0.0003	0.0000	0.0000			
0.09125	0.0056	0.0009	0.0012	0.0003			0.1180*	0.1130*	0.1095*
0.09500	0.0042	0.0006	0.0008	0.0002			0.0992	0.0921	0.0893
0.09875	0.0032	0.0004	0.0006	0.0001			0.0831	0.0748	0.0728
0.10000	0.0029	0.0004	0.0005	0.0000			0.0782	0.0697	0.0679
0.11000	0.0014	0.0001	0.0002				0.0478	0.0391	0.0391
0.12000	0.0006	0.0000	0.0000				0.0287	0.0215	0.0223
0.13000	0.0003						0.0170	0.0116	0.0127
0.14000	0.0001						0.0099	0.0062	0.0072
0.15000	0.0000						0.0058	0.0033	0.0040

Note : The symbol * denotes the value of p for which 1% AOQL is attained.

the per cent of product units accepted on a sampling basis is higher in MLP-2 than in CSP-1 or CSP-2. But when $\delta > 1$, the per cent of product units accepted on a sampling basis is higher in CSP-1 than that in CSP-2 or MLP-2. A comparison of OC(1) curves for different values of δ is provided in Fig. 1.

4. CONCLUSION

When the production process is not under statistical control and at the same time does not follow a scheme of total lack of control, the Markov model is a more realistic model than the Bernoulli model suggested by Dodge[2].

Furthermore, when both the parameters of the Markov model are unknown, the CSP for Markovian scheme using systematic sampling procedure should be used instead of Dodge's CSP which assumes that the production process is under statistical control.

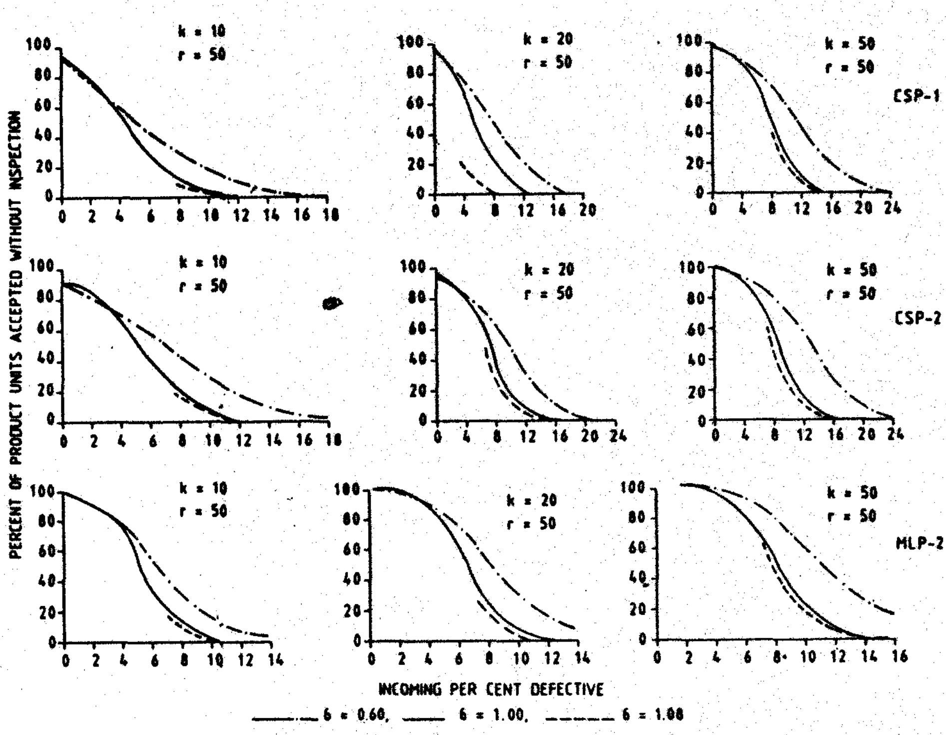

Figure 1. Curves showing the effect of *k* and *r* on OC (1) of the plans.

It may be remarked that when one carries out the data analysis to assess the validity of the Markov model, estimate of the dependence parameter (see Sampath Kumar and Rajarshi[6]) would automatically be available.

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