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Mathematical Modelling of Damage to Aircraft Skin Panels Subjected to Blast Loading

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ABSTRACT

A mathematical model for assessing the damage to an aircraft due to blast from conventional ammunition has been developed. The minimum distance of the point of explosion from the aircraft for its permanent damage for a reference explosion has been obtained depending upon the dimensions (thin plate or thin cylindrical shell) of the structural elements.

1. INTRODUCTION

A military aircraft is subjected to various mechanisms of damage in a warfield. Among these, blast from the high explosive (HE) ammunition is a significant damage mechanism. Since the actual vulnerability of a part of an aircraft depends to a great extent on its area of presentation, the aircraft's structure is by far the largest of the potentially vulnerable items as it consists of nearly 80 per cent of the entire presented area of the aircraft.

It has also been noted[1] that among the various damaging agents, the fragments, incendiary and non-incendiary bullets cause negligible damage to the aircraft structure while the vulnerability due to HE and HE incendiary shells varies. The aircraft structure is highly vulnerable to rods and moderate to highly vulnerable to external blasts.

The chance of survival of an aircraft is influenced by many factors, a major factor among them is the inherent safety or invulnerability of the airframe and its components,

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apart from its flight performance, manoeuvrability, defensive armament, etc. It is the expressed desire of procuring agencies for military aircraft to incorporate the principle of minimum vulnerability in new design concepts within the limitations of overall design requirements[1]. Since an aircraft is usually designed within narrow limits for flight and landing loads, its structure can withstand only small additional loads imposed by weapon effects. In this context an accurate analysis of blast effects becomes necessary for the designers of new aircrafts.

A mathematical model has been developed to estimate the dynamic response of two different structural elements, namely a thin plate with prescribed boundary conditions (such as simply supported or clamped on all edges) and a freely supported thin cylinder subjected to explosive blast pulse. The thin plate model is expected to provide a reasonably accurate analytic simulation of the response of the skin panels, whereas the cylindrical model approximates the dynamic behaviour of entire fuselage structure.

2. THE MATHEMATICAL MODEL

2.1 The Thin Plate Model

We have modified Bauer's formulation[2] for the non-linear response of thin elastic plate to pulse excitations to take into consideration the blast loads with realistic parameters[3]. In order to obtain the results faster and more easily, the perturbation method used by Bauer[2] has been replaced by fourth order Runge-Kutta method. Further a yield criterion, based on von-Mises criterion has been incorporated which indicates the onset of plastic deformation[4]. This may be used to predict the region of permanent damage. The basic equations for large deflection of a thin plate subjected to a time dependent pressure loading are[2] :

$$\nabla^4 = E\left\{\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right\} \tag{1}$$

and

$$\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = \frac{p(x,y,t)}{D} + \frac{h}{D}\left\{\frac{\partial^2 F}{\partial y^2}\cdot\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 F}{\partial x \partial y}\cdot\frac{\partial^2 w}{\partial x \partial y}\right\} \tag{2}$$

where w is the deflection of the plate of thickness h and mass density ρ; $D = Eh^3/12(1-\nu^2)$ is the bending stiffness, E is Young's Modulus and ν is Poisson's ratio. $\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$ is the biharmonic operator and t denotes the time.

F is the Airy's stress function defined by $\frac{\partial^2 F}{\partial y^2} = N_x, \frac{\partial^2 F}{\partial x^2} = N_y$ and $-\frac{\partial^2 F}{\partial x \partial y} = N_{xy}$;

N_x, N_y, N_{xy} being the membrane stresses.

The externally applied load has been taken to be the normally reflected blast pulse (assumed to be uniform over a panel of small dimensions) given by the following relations[3]

$$p(x, y, t) = P_r\left(1 - \frac{t}{t_d}\right)e^{-\alpha t/t_d} \tag{3}$$

$$P_r = \frac{\{8P_o/(P_a + 7)\}\{P_o/(P_a + 1)\}}{\{P_o/(P_a + 7)\}} P_a \tag{4}$$

where P_r, P_o and P_a are the reflected blast pressure for normal incidence, incident blast pressure and ambient pressure respectively, and α and t_d being the wave form parameter and blast pulse duration respectively (assumed to be same as those for the incident blast pulse). The values of the blast parameters may be obtained from the blast chart for conventional weapons or may be generated using Bode-type equations[3]. Using the standard scaling laws, the results may be obtained for any given ammunition.

The problem lies in determining the Airy's stress function F and the plate deflection w satisfying the Eqns (1) and (2) subject to the prescribed boundary conditions. We have taken the panel bounded between consecutive pairs of stringers and ribs as a rectangular plate. Following Bauer[2], the solution has been obtained for both simply supported and clamped plates which may be appropriate for various conditions occurring in the aircraft structure.

The boundary conditions for a simply supported rectangular plate of length a, width b and thickness h are :

$$\begin{aligned} w = 0, \frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} = 0 \text{ at } x = \pm\frac{a}{2} \\ w = 0, \frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2} = 0 \text{ at } y = \pm\frac{b}{2} \end{aligned} \tag{5}$$

whereas for clamped plate, these are given by :

$$\begin{aligned} w = 0, \frac{\partial w}{\partial x} = 0 \text{ at } x = \pm\frac{a}{2} \\ w = 0, \frac{\partial w}{\partial y} = 0 \text{ at } y = \pm\frac{b}{2} \end{aligned} \tag{6}$$

The mid plane displacements u, v in x and y directions respectively are

$$\begin{aligned} u = \int_o^x \left\{\frac{1}{E}\left(\frac{\partial^2 F}{\partial y^2} - \nu\frac{\partial^2 F}{\partial x^2}\right) - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right\} dx \\ v = \int_o^y \left\{\frac{1}{E}\left(\frac{\partial^2 F}{\partial x^2} - \nu\frac{\partial^2 F}{\partial y^2}\right) - \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right\} dy \end{aligned} \tag{7}$$

In accordance with the conditions occurring in aircraft structure, the panel is considered rigidly framed[5]. Hence the edges of the plate have been taken to be immovably constrained, giving further the conditions :

$$u = 0, \frac{\partial^2 F}{\partial x \partial y} = 0 \text{ at } x = \pm\frac{a}{2}$$

$$v = 0, \frac{\partial^2 F}{\partial x \partial y} = 0 \text{ at } y = \pm\frac{b}{2} \tag{8}$$

The exact solution for large deflection in the general case is unknown[5]. Following Bauer's formulation, approximate solutions may be assumed which result in a non-linear ordinary differential equation in an unknown function of time.

2.1.1 Simply Supported Plate

Here the solution has been assumed in the form (satisfying the boundary conditions in Eqn (5))

$$w(x,y,t) = hf(t) \cos\frac{\pi x}{a} \cos\frac{\pi y}{b} \tag{9}$$

To separate the space and time variable, the Airy's stress function is assumed in the form

$$F(x, y, t) = F^*(x, y) f^2(t) \tag{10}$$

Substituting the expression for ω and F from Eqns (9) and (10) in Eqn (1), we obtain

$$\nabla^4 F^* = \frac{Eh^2\pi^4}{2a^2b^2}\left(\cos\frac{2\pi x}{a} + \cos\frac{2\pi y}{b}\right) \tag{11}$$

Using Eqns (7–9), the expression for F^* (x,y) is obtained as

$$F^*(x,y) = \frac{Eh^2}{32}\left[\frac{2\pi^2}{1-\nu^2}\left\{\left(\frac{\nu}{a^2} + \frac{1}{b^2}\right)x^2 + \left(\frac{1}{a^2} + \frac{\nu}{b^2}\right)y^2\right\}\right.$$
$$\left. - \left(\frac{a^2}{b^2}\cos\frac{2\pi x}{a} + \frac{b^2}{a^2}\cos\frac{2\pi y}{b}\right)\right] \tag{12}$$

Now, substituting the values from Eqns (9) and (12) into Eqn (2), the residue is obtained as

$$R = \left[D\pi^4\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2 hf + \rho h^2\ddot{f}\right]\left[\cos\frac{\pi x}{a}\cos\frac{\pi y}{b} + \frac{Eh^4\pi^4}{8} \cdot f^3\right]$$
$$\left[\frac{1}{1-\nu^2}\left(\frac{1}{a^4} + \frac{2\nu}{a^2b^2} + \frac{1}{b^4}\right) + \left(\frac{1}{a^4}\cos\frac{2\pi y}{b} + \frac{1}{b^4}\cos\frac{2\pi x}{a}\right)\right]$$
$$\cos\frac{\pi x}{a}\cos\frac{\pi y}{b} - p(t) \tag{13}$$

Employing Ritz-Galerkin method to solve Eqn (2), we obtain the condition

$$\int_0^{a/2}\int_0^{b/2} R\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}\,dxdy = 0 \tag{14}$$

Carrying out the double integration as indicated in Eqn (14), the equation of motion is obtained as

$$\rho h^2\ddot{f} + \frac{D\pi^4 h}{a^4}\left(1+\frac{a^2}{b^2}\right)^2 f + \frac{Eh^4\pi^4}{8a^4}\left[\frac{\{1+2\nu(a^2/b^2)+(a^2/b^4)\}}{1-\nu^2}\right.$$

$$\left.+\frac{1}{2}\left(1+\frac{a^4}{b^4}\right)\right]f^3 = \frac{16}{\pi^2}p(t) \tag{15}$$

Once this non-linear equation in unknown time function $f(t)$ is solved, the stress function $F(x,y,t)$ and the dynamic deflection of the plate $w(x,y,t)$ can be determined from Eqns (9) and (10) respectively.

2.1.2 Clamped Plate

The approximate solution assumed in this case satisfying the boundary condition in Eqn (6) is

$$w(x,y,t) = hf(t)\cos^2\frac{\pi x}{a}\cos^2\frac{\pi y}{b} \tag{16}$$

The Airy's stress function is again assumed as in Eqn (10) and substituting into Eqn (1), we get

$$\nabla^4 F^*(x,y) = -\frac{Eh^2\pi^4}{2a^2b^2}\left[\cos\frac{2\pi x}{a}+\cos\frac{2\pi y}{b}+2\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{b}\right.$$

$$\left.+\cos\frac{4\pi x}{a}+\cos\frac{4\pi y}{b}+\cos\frac{2\pi x}{a}\cos\frac{4\pi y}{b}+\cos\frac{4\pi x}{a}\cos\frac{2\pi y}{b}\right] \tag{17}$$

Adopting the same procedure as in the previous case, we get the equation of motion as

$$\rho h^2\ddot{f} + \frac{16\pi^4 Dh}{9a^4}\left[3+2\frac{a^2}{b^2}+3\frac{a^4}{b^4}\right]f + \frac{Eh^4\pi^4}{a^4}\left[\frac{\{1+2\nu(a^2/b^2)+(a^4/b^4)\}}{8(1-\nu^2)}\right.$$

$$+\frac{1}{9}\left\{\frac{a^4}{b^4}+\frac{9}{8}+\frac{a^4}{8b^4}+\frac{2(a^4/b^4)}{[1+(a^2/b^2)]^2}+\frac{(a^4/b^4)}{[1+(4a^2/b^2)]^2}\right.$$

$$\left.\left.+\frac{(a^4/b^4)}{[4+(a^2/b^2)]^2}\right\}\right]f^3 = \frac{16}{9}p(t) \tag{18}$$

Once, this equation is solved, the stress function and the deflection of plate are obtained from Eqns (11) and (16) respectively.

2.1.3 Method of Solution

Bauer's original procedure suggests a perturbation technique to solve the non-linear differential equation arising in the equation of motion [Eqns (15) and (18)]. Due to inclusion of cubic terms, this became quite cumbersome and virtually impractical for real applications. Also, the perturbation parameter ε does not appear to be less than unity as claimed by the author, hence the accuracy of Bauer's original solution remains doubtful.

We have proposed a numerical scheme using fourth order Runge-Kutta-Gill method to solve Eqns (15) and (18), hence the deflection of the plate at any instant is immediately obtained and may be plotted very conveniently using a computer. The plate deflection and its velocity are assumed to be zero initially (i.e., $f(0) = f(0) = 0$).

2.1.4 Outset of Plastic Deformation

In order to accommodate the plastic deformation within the present theory, we have proposed that with the increasing intensity of blast pulse, the deformation also increases gradually with accompanying increase in bending moments and membrane stresses. Visualising the outset of plastic deformation as the limiting case of the elastic deformation at the yield point (dynamic yield stress in this case), the elastic relation has been assumed to be valid upto this point.

The yield criterion based on von-Mises criterion is given as[4]

$$\frac{1}{N_p^2}(N_x^2 + N_y^2 + N_xN_y + 3N_{xy}^2) + \frac{1}{M_p^2}(M_x^2 + N_y^2 - M_xM_y + 3M_{xy}^2) - 1 = 0 \quad (19)$$

where $N_p = \sigma_p h$; $M_p = \sigma_p h^2/4$; σ_p being the dynamic yield stress of the plate material. N_k, M_k ($k = x,y,z$) are the membrane stresses and bending moments consistent with earlier notations, and given by

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2}\right)$$

$$M_{xy} = D(1 - v)\,\frac{\partial^2 w}{\partial x \partial y}$$

and
$$N_x = h\frac{\partial^2 F}{\partial y^2}$$
$$N_y = h\frac{\partial^2 F}{\partial x^2}$$
$$N_{xy} = -h\frac{\partial^2 F}{\partial x \partial y} \tag{20}$$

At the centre of the plate (i.e., at the point of maximum deflection), the yield condition Eqn (19) becomes

$$Y_c = \frac{E^2 \pi^4 h^4}{a^2 \sigma_p^2 (1-\nu)^2}\left[\frac{3}{64}(2-\nu')^2 f^4(t) + \frac{1}{9} f^2(t)\right] - 1 = 0 \tag{21}$$

The deformation remains within elastic limit until $Y_c \leqslant 0$, the onset of plastic deformation is indicated at the moment when $Y_c = 0$. The ultimate deflection at this moment may be assumed to be initial values for plastic deformation.

2.1.5 Estimation of Permanent Deflection

Assuming a symmetrical mode of plastic deformation, a method similar to the method of Johnson and Mellor[6] has been used to estimate the plate deflection at the moment when $Y_c = 0$. For a simply supported plate this gives

$$w = \frac{P(V_o a')^2}{6\sigma_p h} \tag{22}$$

where V_o = initial velocity of plate, $a' = min(a,b)$. Similar result may be obtained for clamped plate also.

2.1.6 Comparison with Critical Impulse Criterion of Damage

Sewell and Kinney[7] have proposed a somewhat empirical criterion to predict the failure of aircraft skin panels subjected to blast loading. This states that structural failure under transient loading may be correlated to a critical time duration where the latter is assumed to be one quarter of the natural period of vibration of the structure. The critical impulse is given by

$$I_c = \left(\frac{\rho}{E}\right)^{1/2} h\sigma_p \tag{23}$$

A pressure pulse having a duration of one quarter of the natural period or more, and having an impulse at least equal to I_c will cause the rupture of panel at the attachments. The minimum overpressure required to inflict the damage is given by the ratio of critical impulse to critical time.

2.2 Thin Cylindrical Plate Model

There are many serious difficulties in the analytical modelling for the dynamic response of the fuselage structure of an aircraft due to the following reasons :

(a) The exact pressure distribution around a cylindrical surface is highly non-uniform[8], hence a rigorous evaluation of the structural behaviour is very difficult (at least analytically).

(b) The fuselage structure being stiffened by ribs and stringers, can be more accurately modelled as an orthotropic structure but this is not usable in assessing the permanent damage, as the yield criterion is not known for this type of structure[4].

To overcome such difficulties, some simplifying assumptions have been made :

(a) The pressure distribution has been assumed to be almost uniform around the cylinder. This assumption, although not realistic, gives reasonable estimates for shell behaviour under smoothly varying asymmetric loads such as the one caused by explosive blast[8].

(b) The cylinder has been assumed to be structurally isotropic in order to have a consistency with the yield criterion available at this time.

The equation of non-linear flexural motion for a thin circular cylinder for large deflection is[9,10] :

$$D\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = p(t) + \frac{1}{R}\frac{\partial^2 F}{\partial x^2}\left[\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right] \quad (24)$$

together with the compatibility condition

$$\frac{1}{Eh}\nabla^4 F = -\frac{1}{R}\frac{\partial^2 w}{\partial x^2} + \left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \quad (25)$$

where w is the radial deflection, R is the radius of the shell, the other notations being the same as those for thin plate model. The blast loading is as given by Eqns (3) and (4).

For freely supported boundary conditions (i.e., simple support without axial restraints), the solution is assumed as[9,11] :

$$w = A(t)\sin\alpha_m x \sin\beta_n y + \frac{n^2}{8R}A^2(t)\,(1 - \cos 2\alpha_m x) \quad (26)$$

where $a_m = m\pi/L$, $\beta_n = n/R$. Substituting Eqn (26) in Eqn (25) and integrating, we get

$$F(x,y,t) = Eh\left[\frac{1}{32}\left(\frac{\alpha_m}{\beta_n}\right)^2 A^2 \cos 2\beta_n y + \frac{\alpha_m^2\beta_n^2 n^2 A^3}{4R}\right.$$

$$\left.\left\{\frac{\sin 3\alpha_m x \sin\beta_n y}{(9\alpha_m^2 + \beta_n^2)^2} - \frac{\sin\alpha_m x \sin\beta_n y}{(\alpha_m^2 + \beta_n^2)^2}\right\} + \frac{\alpha_m^2 A}{(\alpha_m^2 + \beta_n^2)^2 R}\sin\alpha_m x \sin\beta_n y\right] \quad (27)$$

Now, substituting the values of w and F in Eqn (24), the expression for the residue R is obtained as in the plate model.

Further the simplifying assumption was made that small perturbations in the loading function may be expressed as

$$p(x,y;t) = q_o(t) \sin \alpha_m x \sin \beta_n y \tag{28}$$

Employing Ritz-Galerkin method to solve for ω, the following condition is obtained

$$\int_{x=0}^{L} \int_{y=0}^{2\pi R} R(x,y,t) \sin \alpha_m x \sin \beta_n y \, dx dy = 0 \tag{29}$$

Carrying out the integrations as indicated in Eqn (29), the equation of motion is obtained as

$$\rho h \ddot{A} + Eh\left[\frac{h^2(\alpha_m^2 + \beta_n^2)}{12(1 - \nu^2)} + \frac{\alpha_m^4}{(\alpha_m^2 + \beta_n^2)^2 R^2}\right]A + Eh\left[\frac{\alpha_m^4}{16} - \frac{\alpha_m^4 \beta_n^2 n^2}{2R^2(\alpha_m^2 + \beta_n^2)^2}\right]A^3$$

$$+ Eh\frac{\alpha_m^4 \beta_n^4 n^4}{16R^2}\left\{\frac{1}{(9\alpha_m^2 + \beta_n^2)^2} + \frac{1}{(\alpha_m^2 + \beta_n^2)^2}\right\}A^5 = q_0(t) \tag{30}$$

which can be solved by Runge-Kutta-Gill method as in the case of thin plate model, using similar initial conditions.

2.3 Outset of Plastic Deformation

The yield criterion given in Eqn (19) is applicable for shells also[4]. The membrane stresses and bending moments for thin shells are obtained exactly in the similar manner as in the case of thin plates. An explicit relation for the yield criterion is obtained using Eqns (26) and (27), which can be conveniently accomodated in the computer programme for the evaluation of the dynamic deflection $w(x,y,t)$.

3. NUMERICAL RESULTS AND DISCUSSIONS

3.1 Thin Plate

For illustration, we have considered a square plate of strong aluminium alloy with 10 cm side and 0.25 cm thickness. The material constants are as : Young's modulus $E = 7.5 \times 10^{11}$ dynes/cm^2, Dynamic yield strength $= 9.7 \times 10^9$ dynes/cm^2, mass density $\rho = 2.8$ gm/cc, Poisson's ratio $\nu = 0.33$.

The resulting pressure pulse on this panel has been taken to be the reflected blast pulse due to detonation of 1 kg TNT. The dynamic deflection of the plate vs time has been plotted in Fig. 1. The minimum distance to cause permanent deformation

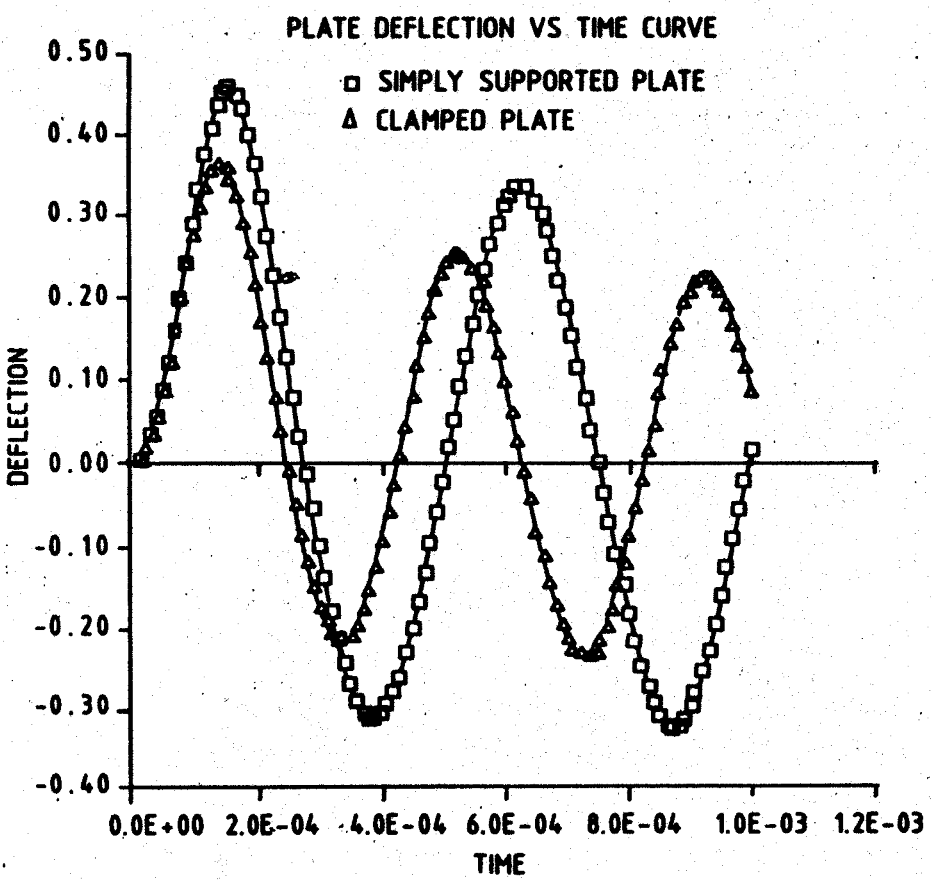

Figure 1. Response of square plate of thickness 0.25 cm and dimensions 10 × 10 cm vs time due to explosion of 1 kg of TNT at a distance of 123 cm from the plate.

is found to be 121 cm approximately for simply supported edges and the corresponding maximum deflection at the centre of the plate is 0.4568 cm. The corresponding deflection of the same place with clamped edges is 0.277 cm.

Taking the maximum initial velocity imparted to the plate as 5200 cm/s (as obtained from Runge-Kutta-Gill algorithm), the maximum plastic deflection is found to be 0.52 cm, which shows good agreement with our model. However, in the case of clamped plate, slight discrepancy is noticed. This might be occurring as the initially assumed deflection profile for clamped plates may not be as good enough as in the case of simply supported plates.

Following the critical impulse criterion, it is found that for this particular panel under consideration critical impulse (I_c) is 4672.0199 dynes/cm^2 s, critical time (t_c) is nearly 2.011 × 10^{-4} s.

Hence the ratio of the critical reflected overpressure to ambient pressure $(p_c/p_a) = 5.4319$ which corresponds to a critical distance of 130 cm from the point of explosion. The corresponding blast duration t_d is 7.5×10^{-4} s $>> t_c$ ensuring a potential damage to the panel under consideration. Keeping in view of the empirical nature of this criterion, this is a reasonably good agreement with the proposed model.

3.2 Thin Shell

Here we have considered a thin shell of the following dimensions: length = 10 cm, radius = 10 cm and thickness = 0.25 cm. The material constants and blast loading data are the same as in the previous model. The non-linear vibrational behaviour is shown in Fig. 2. The minimum distance to cause permanent deflection in this case is found to be 103 cm.

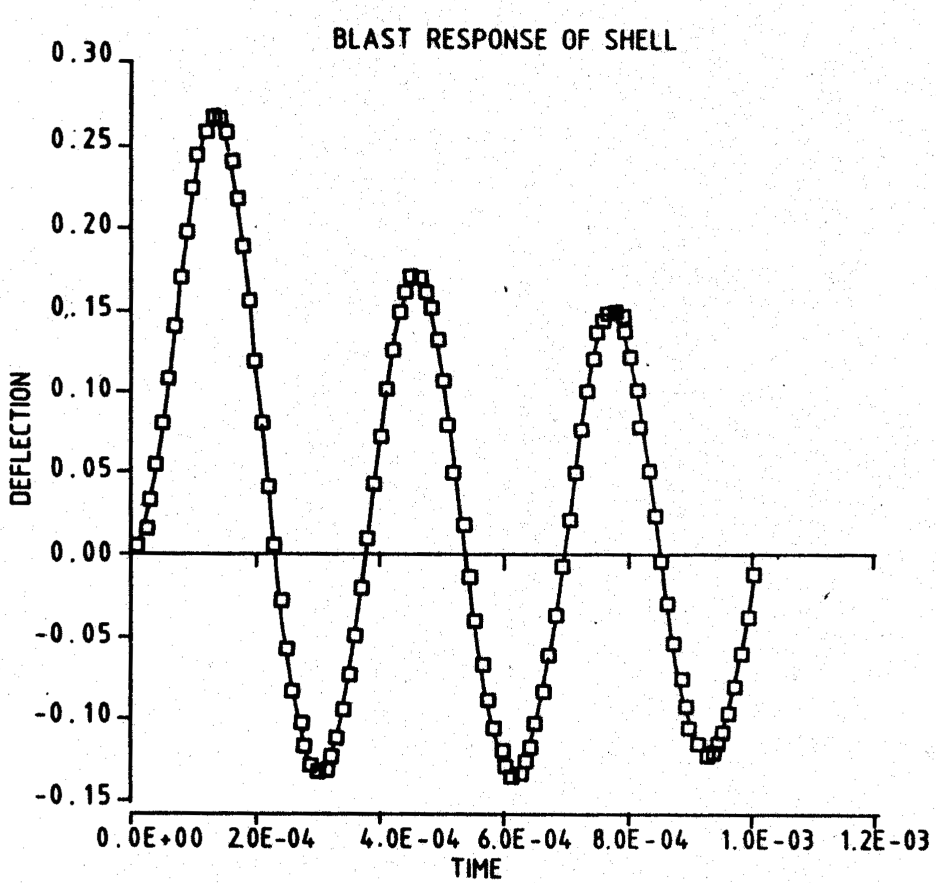

Figure 2. Response of a thin cylindrical shell vs time due to explosion of 1 kg TNT at a distance of 118 cm from the surface of the cylinder.

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