

Some Mathematical Aspects of Spread and Stability of Time-Delay Gonorrhoea

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ABSTRACT

A mathematical model of time-delay gonorrhoea among hetero and homosexuals is presented as a system of first order ordinary coupled integro-differential equations with delayed arguments. A theorem on the positivity of the solutions is proved to establish the feasibility of the proposed model. Further, the only possible diseased equilibrium state has been identified and the stability analysis of such a state for some epidemiological possibilities has been carried out. It has been observed that the impulsive type inflow of infectives into the population maintained the stability of the diseased equilibrium state and is valid even for the exponential type inflows. In contrast to this, instability sets in when one or other of infective inflows is of the gate type.

1. INTRODUCTION

The mathematical study of the epidemic diseases has been a subject of intensive study by both theoretical and experimental scientists working in the areas of population dynamics, social medicine and applied mathematics during the last few decades. Bailey¹, Braun², Cooke and Yorke³ presented mathematical models for the spread of epidemic diseases in the form of simultaneous nonlinear first order ordinary differential equations. Cooke and Yorke³ presented an SIR (susceptibles-infectives-removables) model of gonorrhoea. Later, Braun² described an identical model of gonorrhoea as an application of a threshold theorem on epidemiology relating the infective-susceptible and promiscuous males and females present in the population at

a given time. An exhaustive bibliography on mathematical modelling of gonorrhoea and biological background to the parameters characterising the models have been provided in the monograph by Hethcote and Yorke⁴. The global asymptotic stability of the equilibrium states of the model of Cooke and Yorke³ has been examined by Beretta and Capasso⁵ employing a suitable Lyapunov functional. This paper discusses and examines the stability of the diseased equilibrium state of a model of gonorrhoea, taking the time-delay into account.

A time-delay model of gonorrhoea among hetero and homosexuals has been described in Section 2. The plausibility of the model is established in the form of a theorem. The possible diseased equilibrium state and the secular equation required to describe the stability of such a state have been obtained. In Section 3, the stability analysis of the diseased equilibrium state has been carried out for some typical time-delay kernels characterising the influxes of the infected individuals into the society under consideration. Cases of time-delayed functionals which stabilise or destabilise the diseased equilibrium state are presented as illustrations.

2. THE MODEL

The time-delay mathematical model of gonorrhoea for the spread of the disease proposed in this investigation is characterised by the integro-differential equations :

$$x'(t) = -a_1x(t) + b_1\{c_1(t) - x(t)\} \int_0^{\infty} k_2(z) y(t-z) dz$$

and

$$y'(t) = -a_2y(t) + b_2\{c_2(t) - y(t)\} \int_0^{\infty} k_1(z) x(t-z) dz \quad (1)$$

with the initial conditions

$$x(t)|_{t \in [-z, t_0]} = x_0(t) \geq 0 \quad \text{and} \quad y(t)|_{t \in [-z, t_0]} = y_0(t) \geq 0 \quad (2)$$

where $x(t)$ and $y(t)$ stand for the infective males and females respectively at the instant t of the observation and z is the parameter signifying the time-delay due to gestation, i.e., the delay in time for an infective male or female to contract the disease to a susceptible female or male. Hethcote and Yorke⁴ observed that the time-delay between acquiring infection and becoming infectious in either species would be only one or two days. Further, $c_1(t)$ and $c_2(t)$ are the promiscuous males and females so that $c_1(t) - x(t)$ and $c_2(t) - y(t)$ represent the number for susceptible males and females. In the Eqn (1), ('') denotes derivative with respect to t .

The coefficients a_1, a_2 are the cure-rates while b_1, b_2 are the infective rates of males and females respectively. All these coefficients are naturally positive and would, in general, depend upon the intensity of the spread of the disease due to the interaction

between males and females and various measures adopted from time to time in curing and controlling the disease. As such, these could be, in general, functions of t , $x(t)$ and $y(t)$.

The kernel $k_1(\cdot)$ represents the probability density for a possible delayed union between infective males and susceptible females. Similarly, $k_2(\cdot)$ represents the probability density for a possible delayed union between infective females and susceptible males. These kernels, by definition of the probability density, satisfy the normalisation condition

$$\int_0^{\infty} k_i(z) dz = 1, i = 1, 2 \tag{3}$$

For simplicity, the promiscuous populations $c_1(t)$, $c_2(t)$ and the coefficients a_1 , a_2 , b_1 and b_2 are taken to be constants throughout the present discussion.

It can be noted that the simple model proposed by Braun² can be realised from the above time-delay model by taking $k_1(t) = k_2(t) = \delta(t)$.

2.1 An Important Clinical Observation

The infection is identified in males earlier than in females as the symptoms of the disease in females may be mistaken in the early stages of the diagnosis, even by clinical experts, to be those due to some other gynaecological disorders. Hence the cure-rate of this disease for males is more than that for females, i.e.,

$$a_1 > a_2 \tag{4}$$

2.2 Maximal Contact Rates

The maximal female contact rate F and the maximal male contact rate M are defined as

$$\frac{b_1 c_1}{a_2} = F \quad \text{and} \quad \frac{b_2 c_2}{a_1} = M \tag{5}$$

F may be interpreted as the average number of males that one female infective, before she is completely cured, contracts the disease to susceptible males. Similarly, M is the average number of females that one male infective, before he is completely cured, contracts the disease to susceptible females.

2.3 Positivity of the Solutions

The following theorem on the positivity of the solutions asserts the plausibility of the model proposed.

2.3.1 Theorem

- (a) If the initial infectives $x_0(t)$ and $y_0(t)$ of the model given by Eqn (1) are positive, then $x(t)$ and $y(t)$ are positive for all $t \geq t_0$.

- (b) If $x_0(t)$ is less than c_1 and $y_0(t)$ is less than c_2 , then $x(t)$ and $y(t)$ are less than c_1 and c_2 respectively.

2.3.2. Proof

- (a) Suppose that the statement is false. Let $t^* > t_0$ be the first instant of time at which one of $x(t)$ and $y(t)$ is zero. To start with, let $x(t^*) = 0$. Then

$$\begin{aligned} x'(t^*) &= -a_1 x(t^*) + b_1(c_1 - x(t^*)) \int_0^{t^*} k_2(z) y(t^* - z) dz \\ &= b_1 c_1 \int_0^{t^*} k_2(z) y(t^* - z) dz > 0 \end{aligned} \quad (6)$$

So $x(t)$ is less than zero for t close to and less than t^* , and this contradicts the assumption that t^* is the first instant at which $x(t)$ equals zero. The same contradiction can be noticed for $y(t^*) = 0$. Hence both $x(t)$ and $y(t)$ are positive for $t \geq t_0$.

- (b) Suppose that the statement is false. Let $t^* > t_0$ be an instant at which $x(t) = c_1$ or $y(t) = c_2$. Taking $x(t^*) = c_1$,

$$x'(t^*) = -a_1 c_1 + b_1(c_1 - x(t^*)) \int_0^{t^*} k_2(z) y(t^* - z) dz = -a_1 c_1 < 0 \quad (7)$$

So, $x(t)$ is greater than c_1 for t close to and less than t^* , which contradicts the assumption that t^* is the first time at which $x(t) = c_1$. The similar contradiction can be noticed for $y(t^*) = c_2$. Hence $x(t) < c_1$ and $y(t) < c_2$ for all t .

2.4 Equilibrium States

The equilibrium state of the disease is given by

$$x'(t) = y'(t) = 0 \quad (8)$$

One obvious equilibrium state that can be noted is

$$E_h(0,0) : x = 0 \quad \text{and} \quad y = 0 \quad (9)$$

which corresponds to a perfectly healthy state. The simple model of Braun² has the (non-trivial) diseased equilibrium state

$$x(=x^*) = \frac{b_1 b_2 c_1 c_2 - a_1 a_2}{a_1 b_2 + b_1 b_2 c_1} = \frac{FM - 1}{(b_2/a_2) [1 + (a_2/a_1)F]}$$

and

$$y(=y^*) = \frac{b_1 b_2 c_1 c_2 - a_1 a_2}{a_2 b_1 + b_1 b_2 c_2} = \frac{FM - 1}{(b_1/a_1) [1 + (a_1/a_2)M]} \quad (10)$$

For the existence of this equilibrium state, it is essential that $b_1 b_2 c_1 c_2 > a_1 a_2$, i.e.,

$$FM > 1 \tag{11}$$

In case $FM = 1$, the diseased state $E_d(x^*, y^*)$ reduces to the healthy state $E_h(0, 0)$. It would be natural to assume that the delayed system is operative from the equilibrium state of the simplest model and hence the non-trivial diseased equilibrium state of the time-delay system in Eqn (1) is taken to be the same as $E_d(x^*, y^*)$.

2.4.1 The Secular Equation of the Diseased Equilibrium State $E_d(x^*, y^*)$

Let $u(t)$ and $v(t)$ be small influxes of the diseased males and females into the society when they are in the equilibrium state. The infectious males and females at time t since these influxes are

$$x(t) = x^* + u(t) \quad \text{and} \quad y(t) = y^* + v(t) \tag{12}$$

Substituting Eqn (12) in Eqn (1), $u(t)$ and $v(t)$ satisfy the equations

$$u'(t) = - (a_1 + b_1 y^*) u(t) + b_1 (c_1 - x^*) \int_0^\infty k_2(z) v(t - z) dz \tag{13}$$

$$v'(t) = - (a_2 + b_2 x^*) v(t) + b_2 (c_2 - y^*) \int_0^\infty k_1(z) u(t - z) dz \tag{14}$$

which are the linearised versions of the system in Eqn (1). The secular equation for the system in Eqns (13) and (14) can be obtained as

$$f(\lambda) = \lambda^2 + A_1 \lambda + A_2 - A_3 k_1^*(\lambda) k_2^*(\lambda) = 0 \tag{15}$$

where

$$\begin{aligned} A_1 &= (a_1 + b_1 y^* + a_2 + b_2 x^*) \\ A_2 &= (a_1 + b_1 y^*) (a_2 + b_2 x^*) = a_1 a_2 FM \\ A_3 &= b_1 b_2 (c_1 - x^*) (c_2 - y^*) = a_1 a_2 \end{aligned} \tag{16}$$

It can be directly verified that

$$A_1 > 0, A_2 > 0, A_3 > 0, A_2 > A_3 \text{ and } A_1^2 > 4A_2 \tag{17}$$

Further,

$$k_i^*(\lambda) = \int_0^\infty k_i(z) e^{-\lambda z} dz \tag{18}$$

stands for the Laplace transform of the kernel $k_i(z)$, $i = 1, 2$. For the stability of $E_d(x^*, y^*)$ of the proposed time-delayed system in Eqn (1), all the real roots and the

real parts of all the complex roots, if any, of the secular Eqn (15) are to be negative for all the kernels $k_1(\cdot)$ and $k_2(\cdot)$.

2.5 The Model for Homosexuals

The earlier model relates to the spread of the disease in heterosexuals, i.e., males and females. It is a clinical observation that homosexuals also contract gonorrhoea by physical contacts. The time-delay model for the spread of the disease among homosexuals can be arrived from the system in Eqn (1) by noting that $x(t)$ and $y(t)$ as well as the other parameters in both the populations to be identical. The model is given by the equation

$$x'(t) = -ax(t) + b[c(t) - x(t)] \int_0^{\infty} k(z) x(t-z) dz \quad (19)$$

with the initial condition

$$x(t)|_{t \in [-\tau, 0]} = x_0(t) \geq 0 \quad (20)$$

Also

$$\int_0^{\infty} k(z) dz = 1 \quad (21)$$

Here $x(t)$ is the infective homosexuals at time t , $c(t)$ is the promiscuous population of homosexuals so that $c(t) - x(t)$ is the susceptible population and a , b are the positive constants giving the cure- and infective-rates of the disease in homosexuals. Further, $k(\cdot)$ is a possible delayed union between the susceptible and infected homosexuals. The positivity of the solutions can easily be noticed as in the case of heterosexuals. The only possible equilibrium states are

$$E_h(0) : x = 0 \text{ and } E_d(x^*) : x = x^* = \frac{bc - a}{b} \quad (22)$$

corresponds to the healthy and diseased states respectively. The diseased equilibrium state $E_d(x^*)$ exists only when

$$bc > a \quad (23)$$

A small influx $u(t)$ from the diseased state $E_d(x^*)$ satisfies the linearised form of Eqn (19) given by

$$u'(t) = -bc u(t) + a \int_0^{\infty} k(z) u(t-z) dz \quad (24)$$

and this leads to the secular equation

$$g(\lambda) = \lambda + bc - ak^*(\lambda) = 0 \quad (25)$$

the nature of the roots of which decides the stability of $E_d(x^*)$.

3. SOME INSTANCES OF THE KERNELS WHICH STABILISE/DESTABILISE THE EQUILIBRIUM STATE

The stability of the diseased equilibrium state $E_d(x^*, y^*)$ of the system in Eqn (1) with some special choices of the kernels $k_1(\cdot)$ and $k_2(\cdot)$ characterising the influxes of the male and female infectives is discussed in the following. The results corresponding to homosexuals are also noted in each case.

3.1 Situation I

Let

$$k_i(t) = \delta(t), i = 1, 2 \tag{26}$$

where $\delta(t)$ is impulse function and hence

$$k_i^*(\lambda) = 1, i = 1, 2 \tag{27}$$

An epidemiological interpretation of this situation is a sudden influx of the infected males and females into the population and immediate exit after contacting the infection among susceptibles. It may be because of the immigrant infectives (in both males and females) are weeded-out the moment they are found to be the source of further infection.

The secular Eqn (15) now reduces to

$$f(\lambda) = \lambda^2 + A_1\lambda + A_2 - A_3 = 0 \tag{28}$$

Since $A_1 > 0$; $A_2 > A_3 > 0$ and $A_1^2 > 4 A_2$, both the roots of the Eqn (28) are real and negative and these roots are shown in the Fig. 1. We thus have the following theorem.

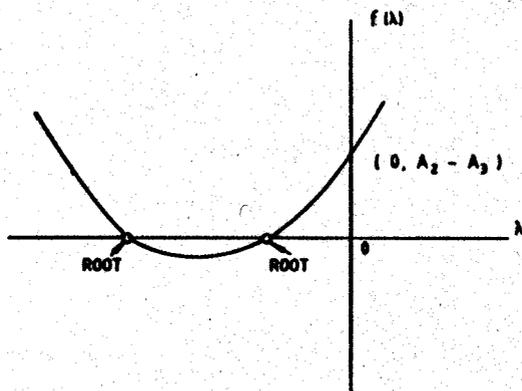


Figure 1. Roots of the Eqn 28.

3.1.1 Theorem I

The diseased state $E_d(x^*, y^*)$ of the system in Eqn (1) with the normalised kernels as given in Eqn (26) is always asymptotically stable.

Remark : For homosexuals, the secular Eqn (25) of $E_d(x^*)$ with $k(t) = \delta(t)$ reduces to

$$g(\lambda) = \lambda + bc - a = 0 \tag{29}$$

the root of which is always negative since $bc > a$ and hence the asymptotic stability of $E_d(x^*)$ follows. Thus the above theorem is true even for the case of homosexuals.

3.2 Situation II

$$k_i(t) = \delta(t - t_i), \quad i = 1, 2 \tag{30}$$

This is the case of a momentary migration of infective males at the time t_1 and (immediate) exit from the population after infecting susceptible females. Similarly, there is a momentary migration and immediate exit of infected females occurring at time t_2 . Here

$$k_i^*(\lambda) = e^{-t_i \lambda}, \quad i = 1, 2 \tag{31}$$

and the secular Eqn (15) now reduces to

$$f(\lambda) = \lambda^2 + A_1 \lambda + A_2 - A_3 e^{-(t_1+t_2)\lambda} = 0 \tag{32}$$

This equation can be rewritten as

$$f(\lambda) = f_1(\lambda) - f_2(\lambda) = 0 \tag{33}$$

where

$$f_1(\lambda) = \lambda^2 + A_1 \lambda + A_2 \text{ and } f_2(\lambda) = A_3 e^{-(t_1+t_2)\lambda} \tag{34}$$

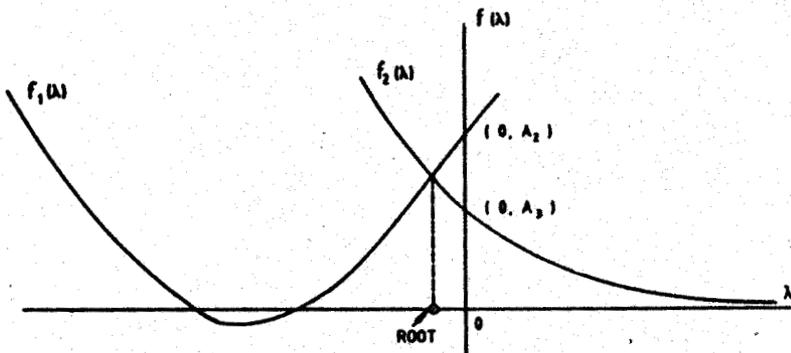


Figure 2. The only possible real negative root of the Eqn 32.

Since $A_1 > 0$, $A_2 > 0$ and $A_1^2 > 4 A_2$, the roots of $f_1(\lambda) = 0$ are real and negative. Further, $f_2(\lambda) = 0$ has no real roots (since the curve $f_2(\lambda)$ versus λ does not cross the λ -axis). The curves $f_1(\lambda)$ and $f_2(\lambda)$ versus λ meet the axis: $\lambda = 0$ at $(0, A_2)$ and $(0, A_3)$ respectively. Since $A_2 > A_3$, the two curves meet at only one point which would be in the second quadrant. Hence the equation $f(\lambda) = 0$ has only one real root and this happens to be negative (Fig. 2).

The nature of the real parts of the complex roots of the Eqn (32) was examined. Taking the Laplace transforms of the linearised system in Eqns (13) and (14) spacing the real valued continuous functions on $[-(t_1 + t_2), \infty]$, we obtain, after some simplification

$$\bar{u}(s) = \frac{b_1(c_1 - x^*) e^{-t_2 s} v(0+) + u(0+)}{s^2 + A_1 s + A_2 - A_3 e^{-(t_1+t_2)s}} \tag{35}$$

where $\bar{u}(s)$ is the Laplace transform of $u(t)$ and $u(0+) > 0$, $v(0+) > 0$. For the system to be asymptotically stable, the real parts of all the poles of $\bar{u}(s)$ should be negative. To establish this, the Nyquist criterion (Freedman, *et al*⁶) was adopted.

Let v_0 be the smallest positive quantity satisfying

$$Im F(iv_0) > 0 \text{ and } Re F(iv_0) = 0 \tag{36}$$

where

$$F(s) = s^2 + A_1 s + A_2 - A_3 e^{-(t_1+t_2)s} \tag{37}$$

the denominator of $\bar{u}(s)$. Hence

$$v_0^2 = A_2 - A_3 \cos \{(t_1 + t_2)v_0\} \text{ and } \frac{A_1}{A_3} > -\frac{\sin \{(t_1 + t_2)v_0\}}{v_0} \tag{38}$$

The inequality in Eqn (38) can be rewritten as

$$\frac{A_1}{A_3(t_1 + t_2)} > -\frac{\sin \{(t_1 + t_2)v_0\}}{(t_1 + t_2)v_0} \tag{39}$$

Restricting v_0 to the interval

$$0 < v_0 < \frac{\pi}{(t_1 + t_2)} \tag{40}$$

Using the hypothesis $A_2 > A_3$, we notice that

$$A_2 - A_3 < A_2 - A_3 \cos \{(t_1 + t_2)v_0\} < A_2 + A_3 \tag{41}$$

i.e., the curves $w = v_0^2$ and $w = A_2 - A_3 \cos \{(t_1 + t_2)v_0\}$ in $w - v_0$ plane intersects on

$$0 < v_0 < \frac{\pi}{(t_1 + t_2)} \quad (42)$$

Since

$$\frac{\sin [(t_1 + t_2)v_0]}{(t_1 + t_2)v_0} < 1 \text{ on } 0 < v_0(t_1 + t_2) < \pi \quad (43)$$

we have from Eqn (41) that the Nyquist criterion is thus satisfied.

$$0 \leq (t_1 + t_2) < (A_1/A_3) \quad (44)$$

We prove that if $f(\lambda) = 0$ has any complex root, its real part can only be negative.

Let

$$t_1 + t_2 = t^* \quad (45)$$

$$\text{Setting } \lambda(t^*) = \mu(t^*) + i\nu(t^*) \quad (46)$$

in Eqn (46) and after separating the real and imaginary parts, we get the transcendental equations

$$\mu^2 - \nu^2 + A_1\mu + A_2 - A_3 e^{-\sigma^*\mu} \cos(t^*\nu) = 0 \quad (47)$$

$$2\mu\nu + A_1\nu + A_3 e^{-\sigma^*\mu} \sin(t^*\nu) = 0 \quad (48)$$

These equations at $t^* = 0$ reduces to

$$\mu^2(0) - \nu^2(0) + A_1\mu(0) + A_2 - A_3 = 0 \quad (49)$$

$$2\mu(0) + \nu(0) + A_1\nu(0) = 0 \quad (50)$$

From the Eqn (50) it can be observed that either $\nu(0) = 0$ or $\mu(0) = -A_1/2$.

In case $\nu(0) = 0$, the roots of the characteristic equation are real and negative and this case has been discussed already.

When $\nu(0) \neq 0$, then $\mu(0) = A_1/2 (< 0)$. Hence when t^* takes positive and sufficiently small values, by continuity and employing Butler's lemma (ref. 6, p. 1003), it can be noted that $\mu < 0$.

Let there exist a $t_0^* > 0$, such that, $\mu = 0$ at $t^* = 0$ at $t^* = t_0^*$ and the solution of Eqns (47) and (48) at $t^* = t_0^*$ be $(0, \nu^*)$. Then from the above equations.

$$A_3 \cos(t_0^*\nu^*) = A_2 - \nu^{*2} \quad (51)$$

$$A_3 \sin(t_0^*\nu^*) = A_1\nu^* \quad (52)$$

Squaring and adding Eqns (51) and (52) we get

$$v^{*4} + (A_1^2 - 2A_2)v^{*2} + (A_2^2 - A_3^2) = 0 \tag{53}$$

From the hypothesis,

$$A_1^2 > 2A_2, \quad A_2^2 > A_3^2$$

and the discriminant

$$A_1^2(A_1^2 - 4A_2) + 4A_3^2 > 0 \tag{54}$$

It is evident that both the roots of this quadratic in v^{*2} are negative. From this, it follows that v^* is not real. This contradicts the existence of real v and so that of t_0^* , hence we have the following theorem.

3.2.1 Theorem II

The diseased state $E_d(x^*, y^*)$ of the system in Eqn (1) with the influx kernels as in Eqn (30) is always asymptotically stable if (i) $A_2 > A_3$ (the condition for the existence of $E_d(x^*, y^*)$), and (ii) $0 \leq (t_1 + t_2) < (A_1/A_3)$, a restriction upon the migrating times t_1 and t_2 of the infectives into the population.

Remark : For the homosexuals in this situation, the secular equation with the kernel $k(t) = \delta(t - t_1)$ reduces to

$$g(\lambda) = \lambda + bc - a e^{-\eta\lambda} = 0 \tag{55}$$

This has the only one possible real negative root. We note that, (i) $bc > a$, and (ii) $0 \leq t_1 < (1/a)$ as the conditions for the diseased state $E_d(x^*)$ to be asymptotically stable.

3.3 Situation III

Let

$$k_i(t) = \begin{cases} \theta_i e^{-\theta_i t}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{56}$$

where $\theta_i (> 0)$ are the mean values of $k_i(\cdot)$, $i = 1, 2$.

This is the case of simultaneous (exponential) decay of both male and female infectives with different expectations. Ultimately, the infection in both the species come down to the level of no infection.

For such a situation,

$$k_i^*(\lambda) = \frac{\theta_i}{(\lambda + \theta_i)}, \quad i = 1, 2 \tag{57}$$

and the secular equation reduces to

$$f(\lambda) = f_1(\lambda) - f_2(\lambda) = 0 \tag{58}$$

where

$$f_1(\lambda) = \lambda^2 + A_1\lambda + A_2 \quad \text{and} \quad f_2(\lambda) = \frac{A_3\theta_1\theta_2}{(\lambda + \theta_1)(\lambda + \theta_2)} \tag{59}$$

It can easily be noticed that

$$f_1(\lambda)_{\min} = f_1\left(-\frac{A_1}{2}\right) = -\frac{1}{4}(A_1^2 - 4A_2) < 0 \tag{60}$$

and

$$f_2(\lambda)_{\max} = f_2\left(-\frac{(\theta_1 + \theta_2)}{2}\right) = -\frac{(4A_3\theta_1\theta_2)}{(\theta_1 - \theta_2)^2} < 0 \tag{61}$$

The Eqn (58) possesses two, three or four real negative roots according as $f_1(\lambda)_{\min}$ is greater than, equal to or less than $f_2(\lambda)_{\max}$ respectively (Fig. 3).

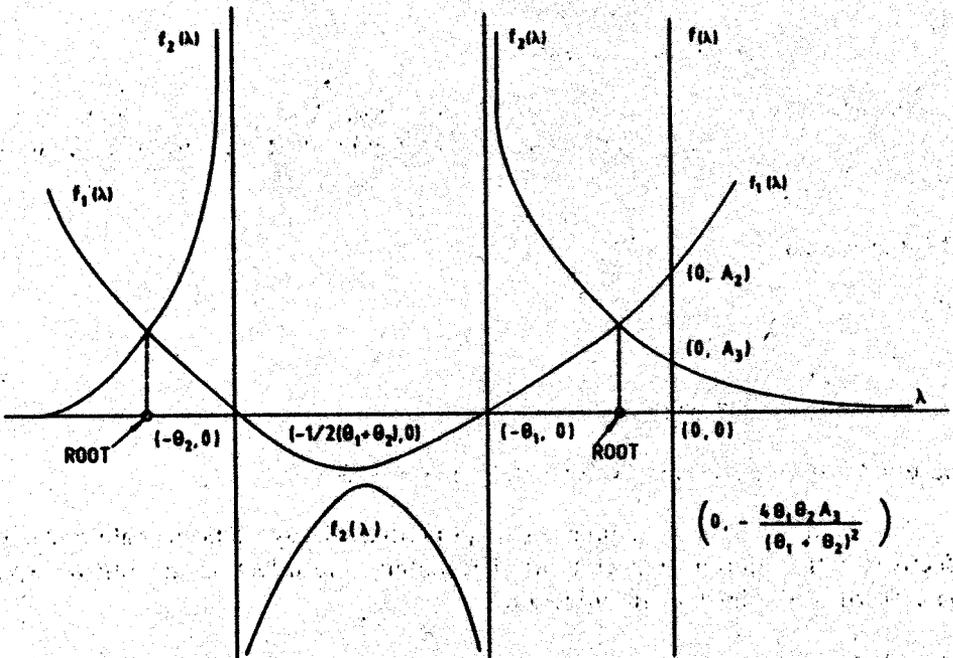


Figure 3(a). The roots of the Eqn 58 when $f_1(\lambda)_{\min} > f_2(\lambda)_{\max}$.

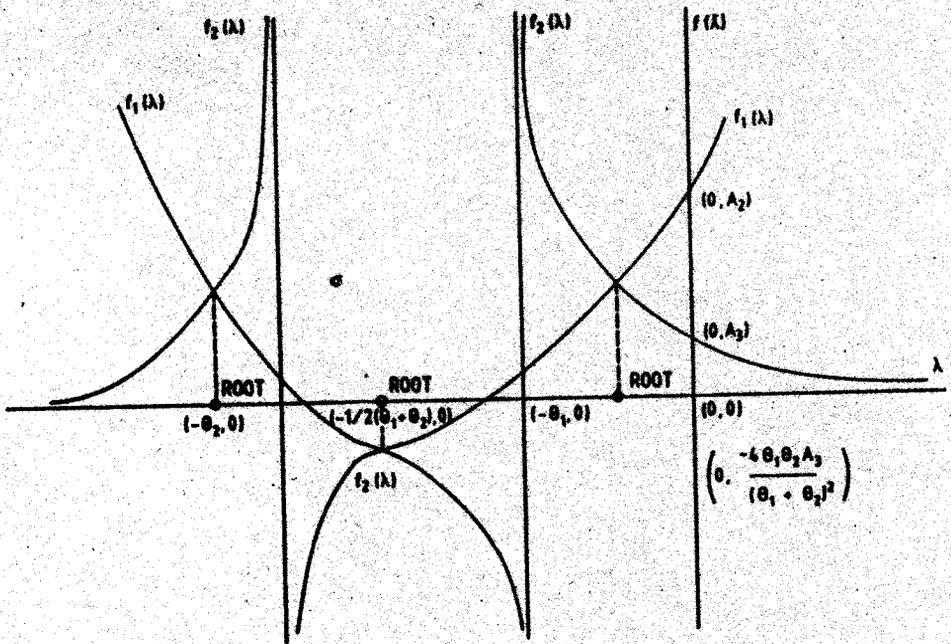


Figure 3(b). The roots of the Eqn 58 when $f_1(\lambda)_{\min} = f_2(\lambda)_{\max}$.

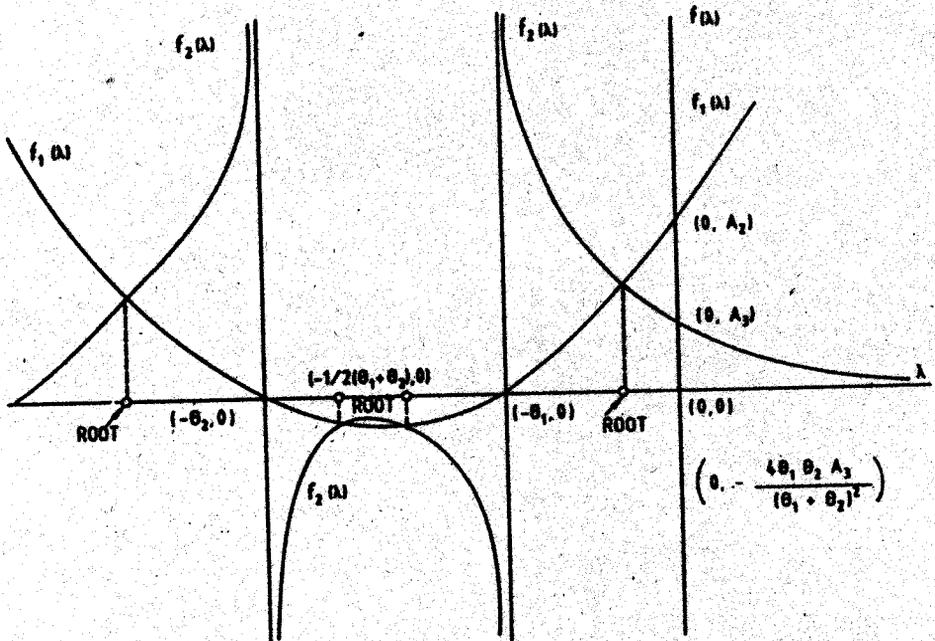


Figure 3(c). The roots of the Eqn 58 when $f_1(\lambda)_{\min} < f_2(\lambda)_{\max}$.

To establish the asymptotic stability, Routh-Hurwitz criterion⁷ was applied to the Eqn (58) after rewriting it as the polynomial equation

$$f(\lambda) = \lambda^4 + B_1\lambda^3 + B_2\lambda^2 + B_3\lambda + B_4 = 0 \quad (62)$$

where

$$B_1 = (\theta_1 + \theta_2 + A_1) > 0 \quad (63)$$

$$B_2 = \theta_1 + \theta_2 + (\theta_1 + \theta_2)A_1 + A_2 > 0 \quad (64)$$

$$B_3 = \theta_2A_1 + (\theta_1 + \theta_2)A_2 > 0 \quad (65)$$

$$B_4 = \theta_1\theta_2(A_2 - A_3) > 0 \quad (66)$$

The Routh-Hurwitz determinants D_1 , D_2 , D_3 and D_4 of Eqn (62) are

$$D_1 = B_1 = (\theta_1 + \theta_2 + A_1) > 0 \quad (67)$$

$$D_2 = B_1B_2 - B_3 = (\theta_1 + \theta_2)(\theta_1 + A_1)(\theta_2 + A_1) + A_1A_2 > 0 \quad (68)$$

$$\begin{aligned} D_3 &= B_1(B_2B_3 - B_1B_4) - B_3^2 \\ &= (\theta_1 + \theta_2)A_1[(\theta_1 + \theta_2)\theta_1\theta_2A_1 + (\theta_1 + \theta_2 + A_1)A_1 \\ &\quad + (\theta_1^2\theta_2^2 + \theta_1\theta_2A_1^2 + A_2^2)] \\ &\quad + \theta_1\theta_2A_2(\theta_1 + \theta_2 + A_1)^2 > 0 \end{aligned} \quad (69)$$

and

$$\begin{aligned} D_4 &= B_4D_3 \\ &= \theta_1\theta_2(A_2 - A_3)D_3 > 0 \end{aligned} \quad (70)$$

Since all the above determinants are positive, the asymptotic stability of the dis-eased equilibrium state follows from the Routh-Hurwitz criterion. Hence the following theorem.

3.3.1. Theorem III

The state $E_d(x^*, y^*)$ of the system in Eqn (1) with the kernels in Eqn (56) is always asymptotically stable.

Remark : For homosexuals, the secular equation with the kernel $k(t) = \theta e^{-\theta t}$, $t > 0$ reduces to

$$g(\lambda) = \lambda + bc - \frac{a\theta}{\lambda + \theta} = 0 \quad (71)$$

All the Routh-Hurwitz determinants of this equation can be noticed to be positive and hence the asymptotic stability of $E_d(x^*)$ follows.

In all the above situations it was observed that the disease can ultimately be levelled-off for the non-trivial equilibrium state of the system. This, however, does not happen always. One case where instability sets in is illustrated below by considering a special type of epidemiologic possibility.

3.4 Situation IV

Let

$$k_1(t) = \delta(t) \text{ and } k_2(t) = \begin{cases} 1/t_2, & t \in [0, t_2] \\ 0, & \text{otherwise} \end{cases} \quad (72)$$

This is an epidemiological situation when both the male and female infectives enter into the population simultaneously contracting the disease to female and male susceptibles. The male infectives are immediately withdrawn while the female infectives stay for a duration of time t_2 before they are withdrawn from the population. Now

$$k_1^*(\lambda) = 1 \text{ and } k_2^*(\lambda) = \frac{1 - e^{-t_2\lambda}}{t_2\lambda} \quad (73)$$

using this we obtain the secular equation

$$f(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda - \frac{A_3}{t_2}(1 - e^{-t_2\lambda}) = 0 \quad (74)$$

Evidently, $\lambda = 0$ is a root of Eqn (74). This shows the destabilisation of the diseased state $E_d(x^*, y^*)$ of the system in Eqn (1). The curves of $f_1(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda - (A_3/t_2)$ and $f_2(\lambda) = - (A_3/t_2) e^{-t_2\lambda}$ have a common point of intersection which lies on the line $\lambda = 0$. Figure 4 shows some of the possible cases where the graphs of $f_1(\lambda)$ and $f_2(\lambda)$ versus λ have a common point on the $f(\lambda)$ -axis. This leads to the following theorem.

3.4.1 Theorem IV

The diseased state $E_d(x^*, y^*)$ of the system in Eqn. (1) with the normalised inflows as in Eqn (72) is always unstable.

It can be easily noticed that, the diseased state always destabilise even if we interchange the kernels of Eqn (72).

Remark : Further, the diseased state $E_d(x^*)$ of homosexuals with the kernel $k(t) = (1/t_1), t \in [0, t_1]$ gives the secular equation

$$g(\lambda) = \lambda^2 + bc\lambda - a + a e^{-t_1\lambda} = 0 \quad (75)$$

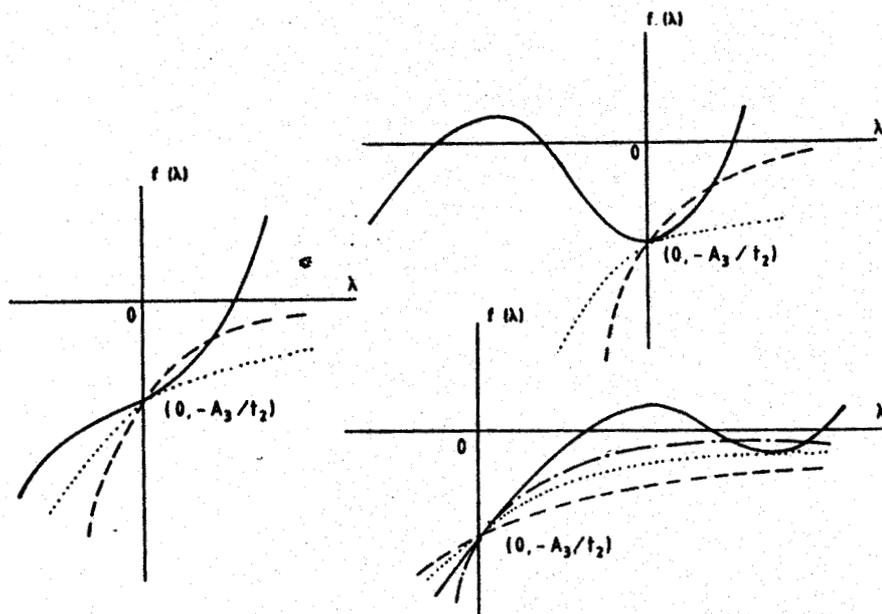


Figure 4. Showing $\lambda = 0$ is a root of the Eqn 74 as the intersection of the curves $f_1(\lambda)$ and $f_2(\lambda)$ vs λ continuous line (—) represents the possible shape of $f_1(\lambda)$ and broken lines (....., - - - - , ———) represents the possible shapes of $f_2(\lambda)$.

By inspection, $\lambda = 0$ is a root and hence the destabilisation of the diseased state $E_d(x^*)$ follows.

4. DISCUSSION

The plausibility and the criteria for stability of the diseased equilibrium state of the proposed model for time-delay gonorrhoea are established. These criteria basically depend upon the time-delay kernels. When both the kernels are either impulsive type or exponential type the diseased equilibrium state will not be destabilised in a migrating time interval, which has been estimated. However, a gate like kernel given in Eqn (72) always destabilises the diseased equilibrium state. This kernel corresponds to the influx of the infective males and females into the population and abrupt removal after their stay for a (non-zero) finite time duration of the diseased contract. This result is noticed to be in contrast with the one for impulsive and exponential type kernels given in Eqns (26), (30) and (56). Investigations on the perfectly healthy state $E_h(0,0)$ yield identical observations.

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