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Pursuit Evasion: An *ab initio* Two-Dimensional Model

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ABSTRACT

A two dimensional pursuit-evasion game is studied using a kinematic model described by algebraic equations of motion. This is the first part of a study aimed at obtaining complete three- dimensional solutions for a full dynamic system including detailed aerodynamic characteristics of the aircraft involved. Comparison with an existing method shows that substantial savings in computer time may be possible using the *ab initio* approach.

1. INTRODUCTION

The problem of pursuit-evasion is one involving two aircraft, one of which is the pursuer and the other the evader[1]. It is the object of the pursuer (*P*) to chase the evader (*E*) and arrive sufficiently close, to enable the necessary action to be performed, typically, release of a weapon. Several authors have analysed situations of varying degrees of realism and using equations and methods of solution of varying complexity[2-11]. In particular De Villiers *et al*[4] have used a stochastic differential game approach. It appeared worthwhile to investigate the possibility of solving the problem using straight forward algebraic equations. Since such an approach requires a full appreciation of the physics of the problem at all stages, it was considered worthwhile to start with a two-dimensional problem. The major advantage of using algebraic equations would be a saving in computational time. This in turn would enable real-time calculations to be performed in a combat situation.

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The two-dimensional case has been solved in this paper as a first step towards a full three-dimensional solution. In order to make it possible to have comparisons with the existing methods, in particular the method of De Villiers *et al*[1] (reduced to two dimensions), typical cases were solved. Identical solutions were obtained by both methods as illustrated in this paper. However, the time taken to solve by using algebraic equation was 1/3 to 1/2 of that using differential equations.

2. THE TWO-DIMENSIONAL MODEL

The simplest case that one may consider is that in which the evader is flying a straight course and the pursuer starts some distance off at a given angle and velocity. If the pursuer is flying right behind the evader in the same line of flight, all one has to do to get close to the evader in the shortest possible time is to use the maximum acceleration. Both the acceleration and the velocity will be limited and this has to be taken into account. Also, when close, the relative velocity of the pursuer should not be too high otherwise the pursuer will have too short a time at close range for necessary action.

In a slightly more practical case where the initial velocities are not coincident in direction, the path of the pursuer has to be much more complex. Obviously the path should take as little time as possible consistent with the availability of acceleration and speed, but should still give sufficient time at close range to enable proper action to be taken. Other factors which complicate the computation include the fact that the pursuer should always remain behind the evader especially when reasonably close. This is necessary, because if the pursuer overtakes the evader, the roles of P and E could get reversed which is not the object of the pursuer. This is another reason why the pursuer's relative velocity at close range will have to be limited. Optimisation of the path may not be simple even in this elementary case.

Another level of complication arises when the evader also takes evasive action (justifying the name). In this case, the pursuer's action will have to be computed at every instant, since it is to be presumed that the evader may have an option of several possible manoeuvres. It is not possible to predict his movements in advance. It is possible to simulate a game on the computer using the same or different capacities for the two aircraft and different initial conditions. Choice of one among several manoeuvres should as far as possible be based on the probability of success (from the point of view of the party making the manoeuvre), and this could mean several calculations to choose an optimum. Again, possible manoeuvres may be limited not only by the capabilities of the aircraft but also by the maximum g endurable by the pilot and corresponding black-out considerations.

One can thus see that the problem is extremely complicated and provides considerable challenge to modelling accurately. Several papers of varying complexity have appeared in the literature[2-11]. One consequence of making a model that is complex to start with is that a physical appreciation of the problem is only too easy to loose sight off. With this in view and to obtain a working knowledge with the equations of motion of aircraft, a two-dimensional model with progressively increasing constraints and realistic extensions is taken up. In this paper three different cases of the model are considered.

Case (a) : The evader travels in a straight course with maximum velocity. We assume that the pursuer has no previous knowledge of the evader's path but extrapolates the future position for *E* and makes point to point decision to decrease the distance between them (Fig. 1).

Case (b) : The evader travels in a straight course for a period of time say t_1 seconds and turns with constant radius (to its right or left) in the rest of its mission. As described in the previous case the pursuer decides its path based on the extrapolated position of the evader (Fig. 2).

Case (c) : Here the path of the evader is not a predetermined one. The evader takes an 'evasive' action based on the previous position of *P*. In other words at each time step both *P* and *E* make a decision based on their positions. *E* takes action to increase the distance between them whereas *P* tries to decrease the distance between them (Fig. 3).

The first two cases are examples of a pure pursuit problem whereas the third case is an example of pursuit evasion of min- max type. The equations of motion are written for a point moving in the *x*-*y* plane, assuming limited acceleration in any direction.

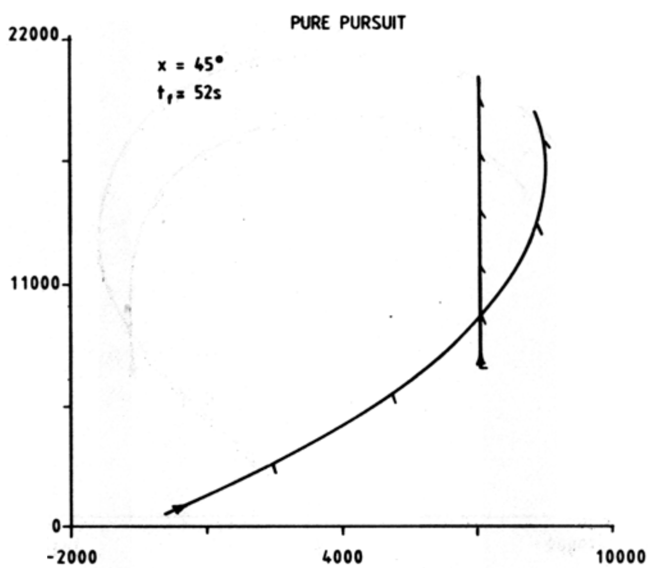

Figure 1(a).

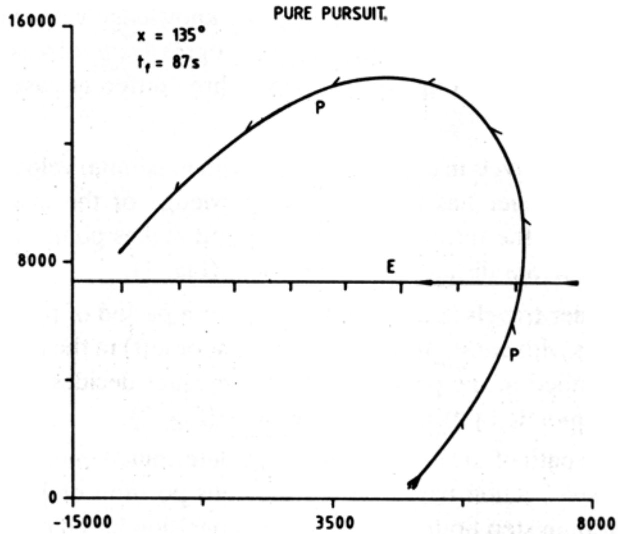

Figure 1. Trajectories of pursuer and evader: evader in straight path.

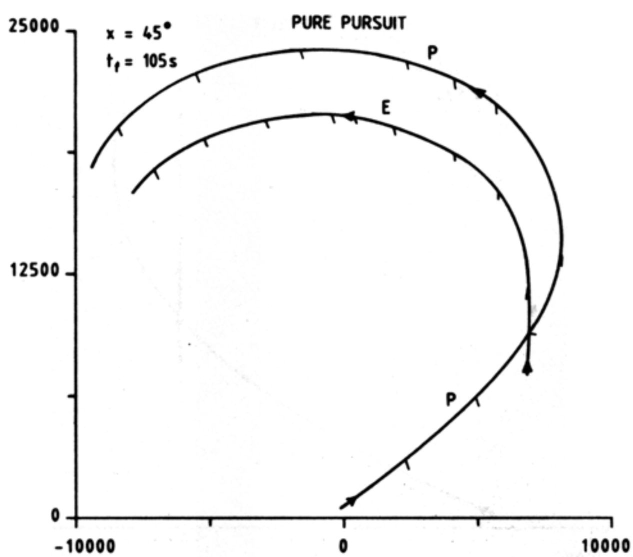

Figure 2(a).

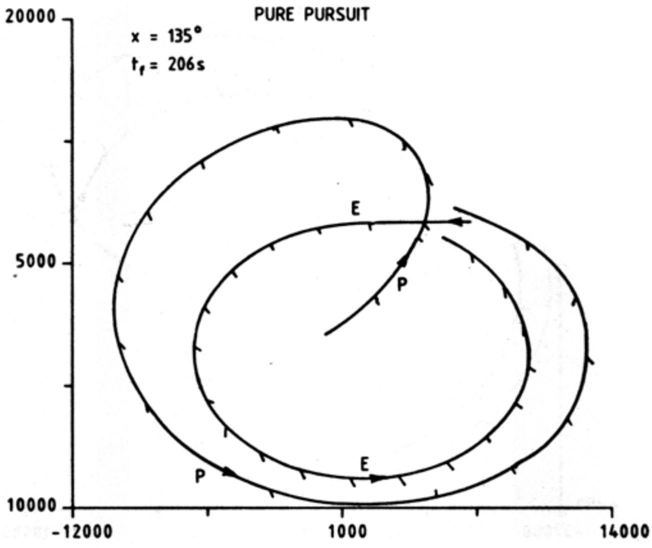

Figure 2. **Trajectories of pursuer and evader: evader in straight path for 20 *s* then turns with a given radius.**

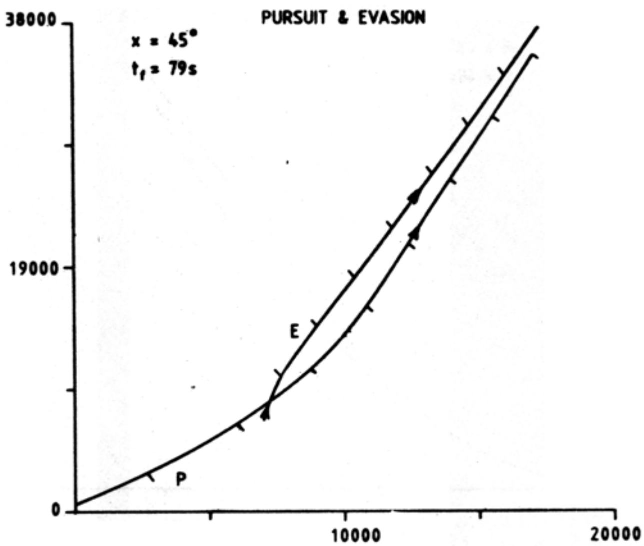

Figure 3(a).

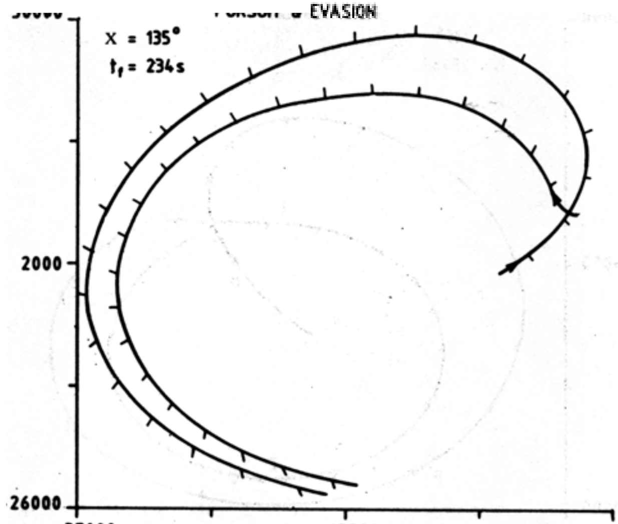

Figure 3(b).

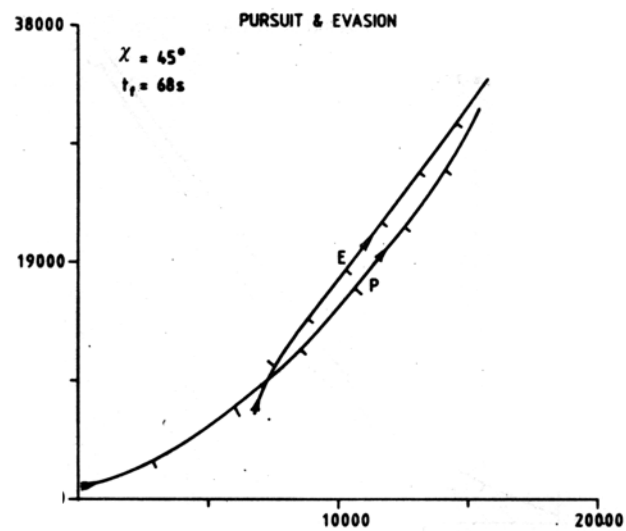

Figure 3(c)

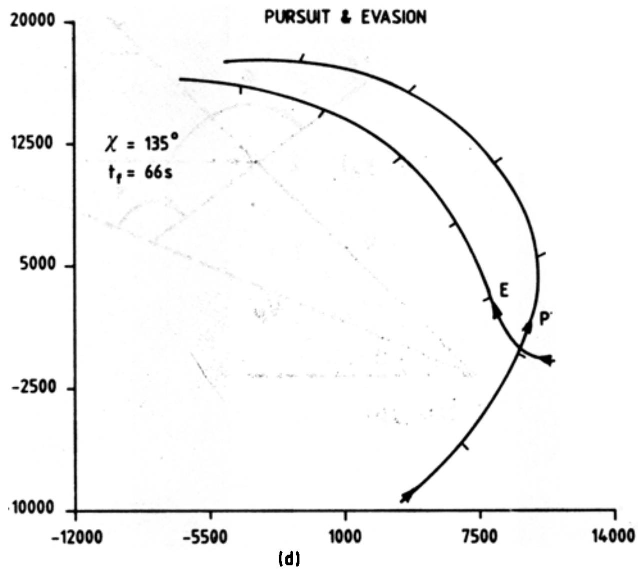

Figure 3. Trajectories of pursuer and evader: both pursuer and evader are manoeuvering.

3. COMPUTATIONAL DETAILS

Consider the motion of P and E in the x-y plane. Let x_p, y_p, v_p, u_p and x_e, y_e, v_e, u_e denote the x-position, y-position, velocity and linear acceleration of P and E respectively. u_p is assumed constant in magnitude. Its direction is chosen to optimise the motion of P. In Case (a), $u_e = 0$. In Case (b), $u_e = 0$ to start with and thereafter u_e is perpendicular to v_e giving a circular path for E. In Case (c), the role of u_e is similar to that of u_p. Let θ and ϕ denote the direction of the velocity vectors of P and E with reference to the x-axis (Fig. 4). Let R denote the distance between P and E at any instant of time,

$$R(t) = \sqrt{(x_p - x_e)^2 + (y_p - y_e)^2} \tag{1}$$

The x-y components for the velocity of P and E are denoted by v_{p_x}, v_{p_y}, and v_{e_x}, v_{e_y} respectively. u_{p_x}, u_{p_y}, u_{e_x} and u_{e_y} denote the acceleration of P and E in x and y directions. Let χ $(= \phi - \theta)$ denote the angle between the velocity vectors v_p and v_e.

Let us assume that, initially P and E are located at (x_p, y_p) and (x_e, y_e) separated by a distance R given by Eqn. (1). Subsequent positions for P and E are calculated as follows :

$$x_{p_1} = x_{p_0} + v_{p_{x_1}} \Delta t \text{ and } y_{p_1} = y_{p_0} + v_{p_{y_1}} \Delta t \tag{2}$$

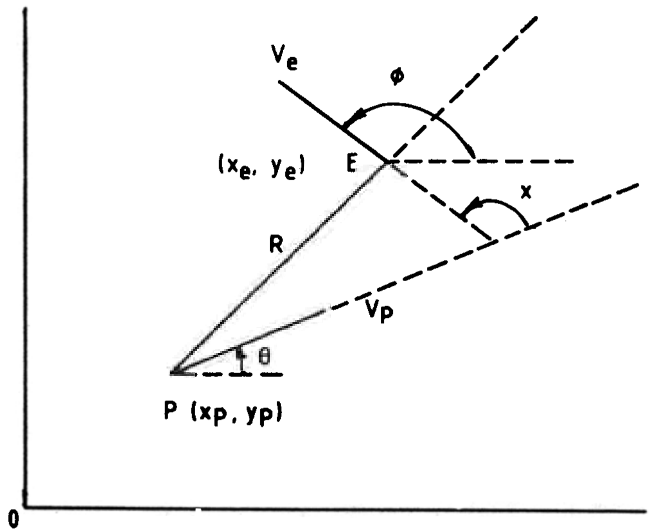

Figure 4. Geometry of the game.

$$v_{p_{x_1}} = v_{p_{x_0}} + u_{p_x} \Delta t; \qquad v_{p_{y_1}} = v_{p_{y_0}} + u_{p_y} \Delta t \tag{3}$$

and

$$u_p = \sqrt{u_{p_x}^2 + u_{p_x}^2} \tag{4}$$

Initially

$$v_{p_x} = v_p \cos\theta; \;\; v_{p_y} = v_p \sin\theta \tag{5}$$

similarly for E

$$x_{e_1} = x_{e_0} + v_{e_x} \Delta t; \;\; y_{e_1} = y_{e_0} + v_{e_y} \Delta t \tag{6}$$

$$v_{e_x} = v_e \cos\phi; \quad v_{e_y} = v_e \sin\phi \tag{7}$$

For the evader, v_{e_x}, v_{e_y} and ϕ remain constant throughout the computation in Case (a).

In general, the value of θ varies as P moves trying to change his direction of velocity in order to get closer to E. The value of θ at any instant is given by

$$\theta \;=\; \tan^{-1} \frac{v_{p_y}}{\text{--}} \tag{8}$$

At every time step the position of E is extrapolated and R is calculated for different combinations of u_{p_x} and u_{p_y} subject to Eqn.(4). The values for u_{p_x} and u_{p_y} are calculated by choosing the direction of acceleration, its magnitude being constant. Thus to get the optimum value for u_{p_x} and u_{p_y}, the directions are chosen in steps and we choose the one for which R is minimum. While optimising the acceleration for P, care is taken to limit the maximum velocity for P. That is, if the v_p resulting from the application of u_{p_x}, u_{p_y} exceeds $v_{p_{max}}$, it is replaced by $v_{p_{max}}$. The computations are carried out till the distance R is less than a predetermined distance R_{min} or the total time of the game exceeds a certain limit.

Computations for Case (b) are carried out in the same way except that for the turn for E after t_1 seconds the position x_e and y_e are calculated as follows :

$$x_e = x_e(t_1) + \frac{v_e \varepsilon}{\omega} (\sin \psi_{t_1} - \sin \psi_0) \tag{9}$$

$$y_e = y_e(t_1) + \frac{v_e \varepsilon}{\omega} (\cos \psi_{t_1} - \cos \psi_0)$$

where

$$\psi_0 = \phi$$

$$\psi_{t_1} = \psi_{t_0} + \varepsilon \, \Delta t \omega$$

$$\omega = \frac{v_e}{r}$$

$\varepsilon = +1$ for left turn or -1 for right turn, and $r =$ radius of turn

Case (c) is the real pursuit evasion case where both pursuer and evader are changing their course. Computations for this case are carried out in similar manner as in Case (a) except that E is also made to change its course. Selection of v_{e_x}, v_{e_y} is made to increase R, that is to enable the evader get away from the pursuer.

Let us denote the linear acceleration of E by u_e and the respective x-y components by u_{e_x} and u_{e_y}. Thus the velocity components v_{e_x} and v_{e_y} are calculated as follows:

$$v_{e_{x_1}} = v_{e_{x_0}} + u_{e_x} \Delta t \tag{14}$$

$$v_{e_{y_1}} = v_{e_{y_0}} + u_{e_y} \Delta t$$

The rest of the computations for position and velocity are carried out as described in Case (a). While computing the accelerations u_{e_x} and u_{e_y} for E, care is taken to limit the velocity to $v_{e_{max}}$. In other words, whenever v_e exceeds $v_{e_{max}}$ it is replaced by $v_{e_{max}}$ for the following step.

4. EXAMPLES

The above calculations are done using Turbo Pascal on an IBM compatible PC. The results are plotted using Turbo Graphics. The data used in the calculations are as follows:

$v_p = 300$ m/s; $v_e = 250$ m/s;
$v_{p_{\max}} = v_{e_{\max}} = 400$ m/s
$R_0 = 10{,}000$ m; $R_{\min} = 2000$ m
$u_p = 20, 30$ m/s^2
$u_e = 20$ m/s^2

The data are common for all the examples worked out except for the value of $v_{p_{\max}}$ which is taken as 500 m/s in Case (c). For Case (a) $u_e = 0$ as E travels with constant velocity. For Case (b) the radius of turn r is taken as 8000 or 4000 m as indicated in the Table 1, and the time t_1 is taken as 20 s. For Case (c) u_e is taken as 20 m/s and $v_{p_{\max}} = 500$ m/s whereas $v_{e_{\max}} = 400$ m/s.

Several examples are worked out varying χ, the angle between v_p and v_e from 0 to 180 in steps of 45 degrees.

The above data are hypothetical and do not pertain to any particualr vehicle.

Table 1 lists the values of the time (t_f) taken to reach the specified minimum range $R_{\min}$(< 2000 m).

It is seen from Table 1 that the programme converges in a time which is variable depending on the initial conditions, viz. initial angle between v_p and v_e, and the relative velocity (both magnitude and direction). Typical cases for the path are drawn in figures as indicated in the table. The following conclusions can be drawn from the table.

(a) As is to be expected the 'closing in time' t_f is least when the two aircraft are travelling towards each other initially. This refers to the case where $\theta = 45$ degrees and $\phi = 225$ degrees which refers to $\chi = 180$ degrees in Table 1. The magnitude of this time (approx. 13 s) also checks with the value one would derive from the change in separation (10,000−2000 = 8000 m) and the relative velocity which in this case is the sum of the velocity (250 + 300 = 550 m/s).

(b) It is found that t_f is larger in Case (b), where the evader is on a non-straight path implying acceleration.

(c) The large values of t_f seen for Case (b) with $u_p = 30$ m/s^2 is due to the small radius of curvature assumed for the evader. Smaller radius of turn implies larger acceleration for the evader.

(d) In Case (c), the values of t_f are generally higher than Case (a) but are reasonable since the acceleration assumed for the evader is moderate.

Table 1. t_f for different initial χ for the three cases

Case	u_p (m/s^2)	u_e (m/s^2)	χ (deg)	t_f* (s)	r (m)	Remarks
(a)	20	0	0	49	Not applicable	Fig.1(a)
			45	52		Fig.1(b)
			90	64		
			135	87		
			180	13		
	30	0	0	46	-do-	
			45	43		
			90	47		
			135	59		
			180	13		
(b)	20		0	102	8000	Fig.2(a)
			45	105	"	Fig.2(b)
			90	160	"	
			135	206	"	
			180	13		
	30		0	353	4000	
			45	364	"	
			90	397	"	
			135	421	"	
			180	13	"	
(c)	20	20	0	101	Not applicable	Fig.3(a)
			45	79		Fig.3(b)
			90	114		
			135	234		
			180	15		
	30	20	0	71	-do-	Fig.3(c)
			45	68		Fig.3(d)
			90	75		
			135	66		
			180	14		

* $R(t_f)$ = 20 % of $R(0)$

5. COMPARISON WITH THE STOCHASTIC MODEL

De Villiers, *et al*[4] have used a moving frame of reference in the derivation of equations of motion of this problem. The same set of equations were reduced to two dimensions and were solved using the initial conditions listed above. The variable calculated is R the distance between P and E as a function of time. This was calculated using the method by the author also and the comparison is shown for two cases, viz. χ = 135 degrees for Case (a) with u_p = 20 m/s^2 and χ = 135 degrees for Case (b) with radius of turn = 8000 m and u_e = 20 m/s^2. Figure 5 illustrates this comparison.

The computational time taken using the algebraic method was 25 s and in the other method was 76 s, using the same computer.

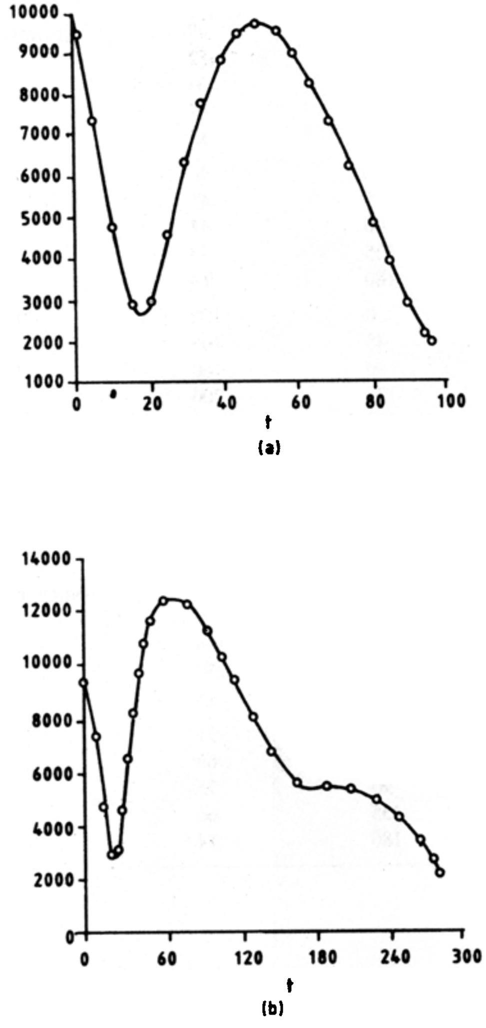

Figure 5. Comparison of solution of Villers, *et al* with the present one.

6. SUMMARY

The problem of $P - E$ is studied *ab initio* in this paper. Equations of motion are derived for the general two-dimensional case. It is shown that it is possible to solve the problem in the general case assuming limited acceleration and realistic velocity limits. Starting from the simplest case in which the evader continues on a straight course, examples are worked out for the case where both pursuer and evader take adaptive action. This study has been undertaken as a first step in the study of three-dimensional problems with aerodynamic force and appropriate constraints. The object of the exercise is to enable a clear understanding of the physical principles involved and is believed that this is achieved in this paper.

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