A Continuum Solution to the Problems of Composite Laminate Analysis

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ABSTRACT

In this paper, a method of solution and some numerical results of free vibration and buckling of composite laminates are presented. All the edges of the laminates are assumed as simply supported. Ritz's method of solution is used with deflection surface expressed in terms of double series in Chebyshev polynomials satisfying the simply supported laminate boundary conditions. The numerical results obtained using the present analysis for isotropic as well as orthotropic plate cases, are compared with the available results. A detailed investigation on the natural frequencies and mode shapes of four-layered, simply supported composite laminates made up of graphite FRP is then undertaken. Numerical results for the first few frequencies and the critical buckling values of symmetric and antisymmetric laminates so obtained, are presented here.

1. INTRODUCTION

Use of laminated composite plates as an important structural element in modern high speed aircraft and missiles has intensified the need to study the vibration and buckling behaviour of such laminates of practically suitable, skew and trapezoidal geometries. It can be found in the literature that considerable attention has been paid to solve the free vibration and buckling problems of isotropic, orthotropic as well as the composite rectangular plates. Free vibration frequencies and the critical buckling coefficients of parallelogrammic, isotropic as well as orthotropic plates with different combinations of boundary conditions, under individual and combined loadings have also received some attention in the past. However, practically no experimental as

well as theoretical results are available in the literature on the frequency and nodal patterns of composite laminates specifically of four sided non-rectangular geometries which are of some relevance in composite construction of the modern aircraft. As pointed out by Leissa[1] it can be noted that very little work has been reported even on the studies of isotropic plates of such geometries. Chopra and Durvasula[2,3] obtained the natural frequencies and mode shapes of simply supported symmetrical and asymmetrical, isotropic trapezoidal plates using Galerkin method. The interesting relationship as the eigenvalues of the isotropic plate being the square of the eigenvalues of the membrane was exploited in their analysis for simply supported plates. However, such relationship ceases to exist for the plate even possessing a significantly low degree of anisotropy. For the study of the behaviours of such plates, there is an imperative need, therefore, to employ a general formulation using simple and economic solution techniques.

In this paper, therefore, free vibration and buckling characteristics of a simply supported general four-sided laminates are investigated while clamped laminate analysis has already been carried out and reported earlier[4]. This laminate geometry is first mapped onto a rectangular domain employing a suitable coordinate transformation[4]. A general formulation[5] based on elastic, higher order shear deformation, thin-layered composite plate theory is used here. In this analysis, trigonometric functions are used for in-plane displacements (U, V) and rotations of middle plane (β_x, β_y), whereas a double series, expressed in Chebyshev polynomials is used here as the assumed deflection surface for out-of-plane displacements. Numerical results obtained for the isotropic plate cases using the present analysis are compared first with previously published results.

2. MATHEMATICAL ANALYSIS

The governing matrix equation is obtained here by using the variational method of Ritz, employing the admissible functions for the displacements as

$$U = \sum_{m=0}^{M} \sum_{n=0}^{N} 2A_{mn} \sin m\pi\xi \sin n\pi\eta \tag{1}$$

$$V = \sum_{m=0}^{M} \sum_{n=0}^{N} 2B_{mn} \sin m\pi\xi \sin n\pi\eta \tag{2}$$

$$\beta_x = \sum_{m=0}^{M} \sum_{n=0}^{N} 2C_{mn} \sin m\pi\xi \sin n\pi\eta \tag{3}$$

$$\beta_y = \sum_{m=0}^{M} \sum_{n=0}^{N} 2D_{mn} \sin m\pi\xi \sin n\pi\eta \tag{4}$$

$$W = \sum_{m=0}^{M} \sum_{n=0}^{N} E_{mn} X_m(\xi) Y_n(\eta) \tag{5}$$

where $X_{m(\xi)}$, $Y_{n(\eta)}$ are the generated functions in Chebyshev polynomials satisfying the simply supported laminate boundary conditions and are obtained from the equations as given by Chakrabarti and Joga Rao[6], and Kamal and Durvasula[7] which are reported here for the sake of completeness.

Using the properties of Chebyshev polynomials as

$$w_n = \sum_{n=0}^{N}{}' A_n T_n(\zeta); \quad \zeta^r T_r(\zeta) = 2^{-r} \sum_{i=0}^{r}{}' (\ ) T_{n-2r+2i}(\zeta)$$

$$\zeta = 4\xi - 3$$
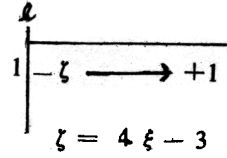
and $\quad A_{n-1,\,(s+1)} = A_{n+1,\,(s+1)} + 2n A_{n,s}$

Differential equation for a cylindrically orthotropic plate is given as

$$\sum_{n=0}^{N-4}{}' [\alpha\{A_{n,\zeta\zeta\zeta\zeta} + (3/2)(A_{n-1,\zeta\zeta\zeta\zeta} + A_{n+1,\zeta\zeta\zeta\zeta}) + (3/2)(A_{n-2,\zeta\zeta\zeta\zeta}$$

$$+ 2A_{n,\zeta\zeta\zeta\zeta} + A_{n+2,\zeta\zeta\zeta\zeta}) + (1/8)(A_{n+3,\zeta\zeta\zeta\zeta} + 3A_{n+1,\zeta\zeta\zeta\zeta}$$

$$+ 3A_{n-1,\zeta\zeta\zeta\zeta} + A_{n-3,\zeta\zeta\zeta\zeta}) + 2(A_{n-1,\zeta\zeta\zeta} + A_{n,\zeta\zeta\zeta} + A_{n+1,\zeta\zeta\zeta})$$

$$+ (1/2)(A_{n-2,\zeta\zeta\zeta} + 2A_{n,\zeta\zeta\zeta} + A_{n+2,\zeta\zeta\zeta})\}$$

$$- \{A_{n,\zeta\zeta} + (1/2)(A_{n-1,\zeta\zeta} + A_{n+1,\zeta\zeta})\} + A_{n,\zeta}] T_n(\zeta)$$

$$= \sum_{n=0}^{N}{}' \lambda'\{(5/32)A_n + (15/138)A_{n+1} + (3/64)A_{n+2} + (1/128)A_{n+3}$$

$$+ (1/128)A_{n-3} + (3/64)A_{n-2} + (15/128)A_{n-1}] T_n(\zeta) \tag{6}$$

By virtue of axisymmetry of the geometry and boundary conditions, it follows that at the centre the conditions for such plates (at $\zeta = -1$) are

$$\sum_{n=0}^{N}{}' A_n T_n(\zeta) = 0 \tag{7}$$

$$\sum_{n=0}^{N-3}{}' A_{n,\zeta\zeta\zeta} T_n(\zeta) = 0 \tag{8}$$

For a simply supported plate edge conditions to be imposed (at $\zeta = -1$) are

$$\sum_{n=0}^{N}{}' A_n T_n(\zeta) = 0 \tag{9}$$

$$\sum_{n=0}^{N-2}{}' \{A_{n,\zeta\zeta} + (1/2)(A_{n+1,\zeta\zeta} + A_{n+1,\zeta\zeta})\} T_n(\zeta)$$

$$+ \beta \sum_{n=0}^{N-1}{}' A_{n,\zeta} T_n(\zeta) = 0 \tag{10}$$

where

$$\lambda' = \rho\omega^2/D'_{22} \text{ for } \xi = \pm 1; \quad \alpha = D'_{11}/D_{22}$$

$$\text{for } \xi = \pm 1$$

$$\beta = \beta_1 = (D'_{33}D'^2_{12} - D'_{13}D'_{23})/(D'_{11}D'_{33} - D'_{13}D'_{13})$$

$$\text{for } \xi = \pm 1$$

$$= \beta_2 = (D''_{33}D''_{12} - D''_{13}D''_{23})/(D''_{22}D''_{33} - D''_{23}D''_{23})$$

$$\text{for } \eta = \pm 1$$

Equations (6-10) are of the form

$$[C_{ij}][A_j] - \lambda'[A_j] = 0$$

which can be solved using eigenvalue subroutines and $\{ A_j \}$ can be obtained for various values of λ'.

The admissible function which can be used for the displacement W is as

$$X_m(\xi) = \sum'_{I=0} A_I T_I^*(\xi)$$

$$Y_n(\eta) = \sum'_{j=0} A_j T_j^*(\eta)$$

where A_i and A_j are the coefficients of the shifted Chebyshev polynomials T_i^* and T_j^* respectively. The coefficients A_{mn}, B_{mn}, C_{mn}, D_{mn} and E_{mn} in Eqns (1-5) are then determined on minimising the Lagrangian L of the system as

$$(\partial L/\partial A_{mn}) = 0; \ (\partial L/\partial B_{mn}) = 0; \ (\partial L/\partial C_{mn}) = 0; \ (\partial L/\partial D_{mn}) = 0; \tag{11}$$

$$(\partial L/\partial E_{mn}) = 0$$

Equation (11) then results in a set of linear, homogenous, simultaneous algebraic equations of the form

$$\left[[S] - k_n^2[M] \right] \begin{bmatrix} A_{rs} \\ B_{rs} \\ E_{rs} \\ C_{rs} \\ D_{rs} \end{bmatrix} = 0$$

The elements of stiffness matrix $[S]$ are expressed here[7] in terms of stretching, bending-stretching and bending stiffness matrices, mapped onto a square domain employing the mapping function $[A_q]$ as

$$[A_q] = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}$$

where $\quad \alpha_1 = -a \cos \psi_1/[\sin \psi_1(b_2 \cos \psi_2 - b_1 \cos \psi_1)$

$$+ \cos \psi_1\{b_1 \sin \psi_1 - (a - b_2 \sin \psi_2)\}]$$

$$\alpha_2 = a \sin \psi_1[\sin \psi_1(b_2 \cos \psi_2 - b_1 \cos \psi_1)$$

$$+ \cos \psi_1 \{b_1 \sin \psi_1 - (a - b_2 \sin \psi_2)\}] \tag{15}$$

$$\alpha_3 = a(b_2 \cos \psi_2 - b_1 \cos \psi_1)/[b_1 \sin \psi_1(b_2 \cos \psi_2 - b_1 \cos \psi_1)$$

$$+ b_1 \cos \psi_1\{b_1 \sin \psi_1 - (a - b_2 \sin \psi_2)\}]$$

$$\alpha_4 = a\{b_1 \sin \psi_1 - (a - b_2 \sin \psi_2)\}/[b_1 \sin \psi_1(b_2 \cos \psi_2$$
$$- b_1 \cos \psi_1) + b_1 \cos \psi_1\{b_1 \sin \psi_1 - (a - b_2 \sin \psi_2)\}] \tag{17}$$

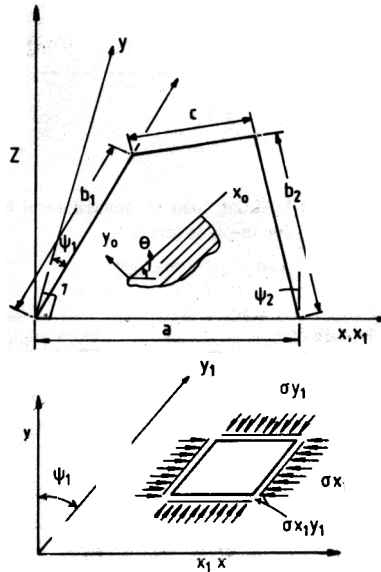with $\quad a, b_1, b_2, \psi_1, \psi_2$ as defined in Fig. 1.

Figure 1. General quadrilateral laminate showing the coordinate axes and in-plane stress system in terms of oblique components.

3. NUMERICAL RESULTS AND DISCUSSION

Numerical calculations for the critical buckling values, natural frequencies and the corresponding modes for the four-layered antisymmetric laminates with simply supported edges are obtained. These results are shown graphically as a function of the ratio of the length of two sides, top and base of each laminate.

Natural frequencies and the critical buckling values of composite laminates are obtained as dimensionless parameters. The buckling parameters for the example cases are reported in Tables 1 and 2. The variation of the values of the frequency parameters of these laminates are shown in Figs. 2 and 3. From these figures, it can be noticed that as the general four-sided laminate configuration becomes a square, the pair of vibration modes become degenerate. However, a slight asymmetry into a system is found to remove this degeneracy. This feature of the laminates is in conformity with the earlier observations on the degeneracy of these curves for isotropic plate cases[10]. Similarly, the frequency crossings between pairs of modes belonging to skew symmetric and skew antisymmetric cases are also apparent. It can be noticed that such crossings are more pronounced amongst the higher modes for these laminates. In the cases of laminates considered, it is also noticed that the frequency and buckling parameter

Table 1. Comparison of critical buckling load of 0°/90°/90°/0° ply, square simply supported composite laminate with in-plane stress σ_x

[0°/90°/0°/90°]

| E_1/E_2	Present solution	Phan & Reddy[8]	3-D elasticity[9]	.	CPT*
40.0	23.804	23.34	22.88		36.16
10.0	10.171	9.774	9.762		11.492

Note : $a/t_0 = 10$; $G_{12}/E_2 = 0.6$: $G_{23}/E_2 = 0.5$; $v_{12} = 0.25$

* Classical plate theory

Table 2. Convergence study for critical buckling load of quadrilateral laminate with simply supported boundary $(S_1 - S_1 - S_1 - S_1)$ for in-plane stress σ_y.

$[45^0/-45^0/45^0/-45^0]$

M	N	Matrix size	Nature of $(m+n)$	0	Skew angle ψ_1(degrees) 15	30	60
1	1	(5 × 5)	E	67.54	80.18	132.48	1225.26
1	2	(10 × 10)	E	67.54	80.18	132.48	1225.26
2	2	(20 × 20)	E	67.54	75.18	110.74	925.22
3	3	(30 × 30)	E	67.54	74.93	107.61	847.44
2	4	(40 × 40)	E	67.54	74.64	107.01	845.90
3	3	(45 × 45)	E	67.54	74.25	98.47	637.99
3	4	(60 × 60)	E	67.54	74.64	98.39	592.36
4	4	(80 × 80)	E	67.54	73.42	97.16	489.35
4	5	(100 × 100)	E	67.54	73.33	96.59	465.94
5	5	(125 × 125)	E	67.54	73.31	96.46	466.02

Note : $a/b = 1.0$; $R_y = \sigma_y b^2/t^2 E_2$; $E_1/E_2 = 40$; $v_{12} = 0.25$; $G_{12}/E_2 = 0.5$
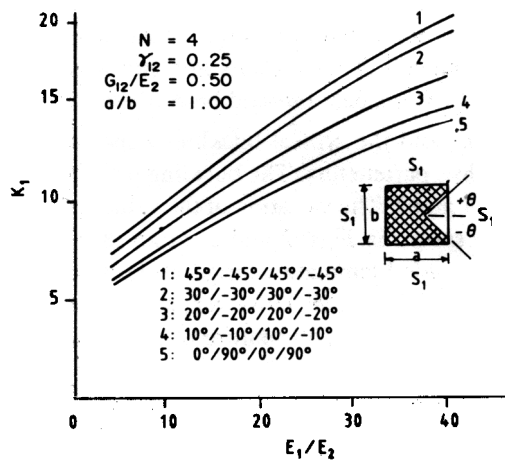
Figure 2. Variation of frequency parameter with material properties.

values as expected, decreases almost exponentially with the ratio of top and base length (c/a) of the composite laminates. The nodal pattern of a representative case is presented in Fig. 4.
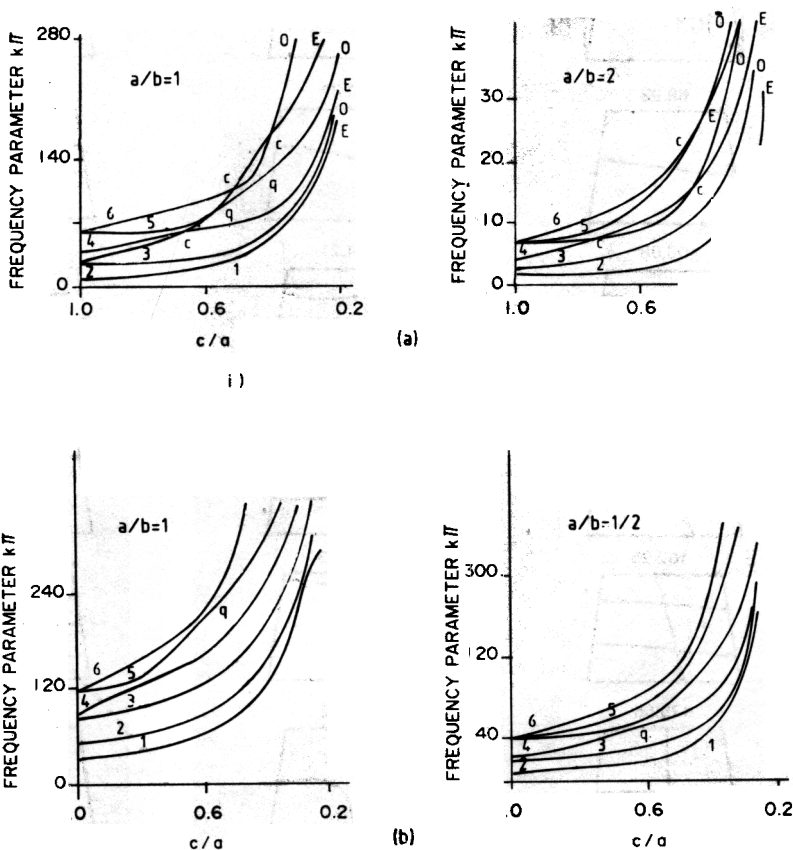

Figure 3. Variation of frequencies with ratios of (a) two parallel sides of simply supported cross-ply graphite fibre reinforced plastic symmetric trapezoidal laminates with (i) $a/b = 1$, and (ii) $a/b = 2$ respectively, and (b) sides of top and base of simply supported antisymmetric-ply unsymmetrical trapezoidal laminates with (i) $a/b = 1$, and (ii) $a/b = 1/2$ respectively.

4. CONCLUSIONS

In this paper, a method of solution and the numerical results of free vibration and the buckling parameter values for four-sided non-rectangular composite laminates using variational method of Ritz employing Chebyshev series are presented. While the frequencies and the buckling values for several different configurations are obtained, the results for only a few illustrative cases are reported here for the simply supported laminates. It is believed that apart from providing the source of new data for comparison with other solution techniques, the important features such as the
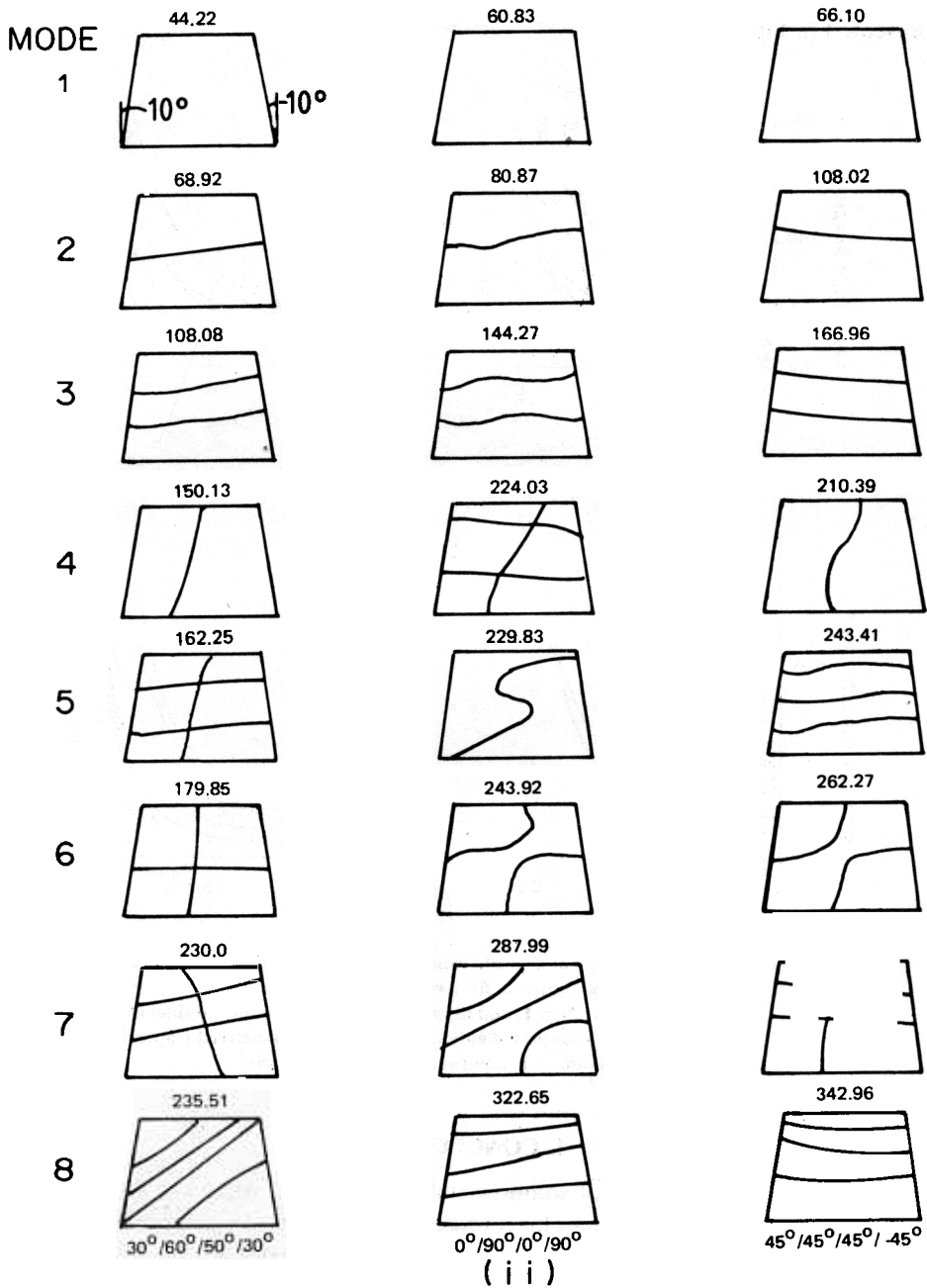
Figure 4. Nodal patterns of (i) symmetric-ply, (ii) cross-ply, and (iii) antisymmetric-ply, graphite fibre reinforced plastic, simply supported symmetric trapezoidal laminates.

occurrence of crossings and quasi-degeneracy of frequency curves, practically for the first time are observed for composite laminates during this study.

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