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Sensitivity of Bayes Estimates of Reliability and Reliable Life to a Non-Standard Prior

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ABSTRACT

Robustness of Bayes estimate of reliability function and the reliable life by employing Edgeworth-gamma class of priors for the unknown mean and precision of normal population of failure times, have been investigated. The effect of a moderately non-normal prior for Martz and Waller data has been noted to be insignificantly small.

1. INTRODUCTION

Reliability analyses evaluate the performance of sophisticated devices which are designed, developed, and implemented for space explorations, military applications, and a variety of commercial uses. The sampling theory methods are found to be inappropriate to incorporate known facts regarding the reliability of the current hardware in an attempt to improve quality of the reliability estimates of the new design. Bayesian approach to reliability estimation utilises all available information—both objective test data and analyst's subjective information—in a most effective manner. Martz and Waller[1] provides a comprehensive reference devoted to Bayesian methods for reliability analyses.

A major criticism of Bayesian reliability analysis is the use of a single prior distribution which is based on degree of belief of an analyst. Prior distributions can never be quantified or elicited exactly (i.e., without error) in a finite time. Any analysis, therefore, based on a convenient prior is questionable. Berger[2,3] extensively reviews robust Bayesian viewpoint in which the question of sensitivity of Bayesian inferences/decisions to slight changes in the prior distribution, is discussed in detail. A reasonable approach is to consider a class of plausible priors which are in the neighbourhood of a specific assessed approximation to the true prior and examine the robustness of the inference/decision with respect to this class of prior distributions.

Martz and Waller[1] discussed Bayes estimation of reliability and reliable life for both non-informative and natural conjugate prior distributions for unknown mean and precision of normally distributed failure time. In this paper, use of Bansal's approach[4] will be made by considering a class of Edgeworth-gamma (EG) priors to study robustness of Bayes estimates of reliability and reliable life to moderate amount of non-normality in the prior distribution of the unknown mean of a normal distribution for which precision is unknown. Bansal[4-7], Chakravarti and Bansal[8], and Bansal and Sinha[9], have used Edgeworth series distribution (ESD) class of priors to investigate effect of non-normal priors for the unknown normal mean on Bayes decisions, forecasts, and sampling inspection plans.

2. PREDICTIVE DENSITY FUNCTION

Suppose that $t=(t_1, t_2,\ldots, t_n)$ is a random sample of n complete failure times available from a $N(\theta, \psi)$ population with unknown mean θ and precision $\psi(> 0)$. The likelihood function is

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$$L(\theta, \psi) = (\psi/2\pi)^{1/2\, n} \exp[-(\psi/2)\,\Sigma(t_i - \theta)^2]$$

$$= (\psi/2\pi)^{1/2\, n} \exp[-(\psi/2)\{n(\theta - \bar{t})^2 + \Sigma(t_i - \bar{t})^2\} \quad (1)$$

The natural conjugate prior for θ and ψ is a normal-gamma (NG) distribution[10] To study the robustness of reliability and reliable life to slight deviation from such a prior, the class of EG priors can be considered[4]. Here, taking the conditional prior density of θ, given $\psi = \psi$, as ESD

$$\xi(\theta \mid \psi) = (\tau\psi/2\pi)^{1/2} \exp[-\tfrac{1}{2}\psi\tau(\theta - \mu)^2]\, H(\theta) \quad (2)$$

where

$$H(\theta) = 1 + \tfrac{1}{6}\lambda_3 H_3\{\sqrt{\tau\psi}\,(\theta - \mu)\} + \tfrac{1}{24}\lambda_4 H_4\{\sqrt{\tau\psi}\,(\theta - \mu)\} + \tfrac{1}{72}\lambda_3^2 H_6\{\sqrt{\tau\psi}\,(\theta - \mu)\} \quad (3)$$

and $H_k(.)$ is polynomial of degree k, λ_3 and λ_4 are respectively, the measures of skewness and kurtosis. It is known that within Barton and Dennis[11] region, an ESD is a proper unimodal probablity density function (pdf)[12]. Assume that the marginal prior density of the unknown precision is gamma with parameters $a > 0$ and $\beta > 0$

$$\xi(\psi) = \beta^a \psi^{a-1} \exp(-\beta\psi) / \Gamma(a),\ \psi > 0 \quad (4)$$

so that the joint prior of θ and ψ is

$$\xi(\theta, \psi) = \xi(\theta \mid \psi)\, \xi(\psi)$$

$$= \left(\frac{\tau}{2\pi}\right)^{1/2} \beta^a \psi^{a-1/2} \exp[-\psi\{\beta + \frac{\tau}{2}(\theta - \mu)^2\}]\, H(\theta)/\Gamma(a) \quad (5)$$

On using the posterior density[4] of (θ, ψ) the identity

$$(\theta - x)^2 + \tau'(\theta - \mu')^2 = \tau''(\theta - \mu_1)^2 + \tau'(x - \mu')^2/\tau'' \quad (6)$$

and the integral

$$\frac{1}{\sqrt{2\pi}} \int \int^{\infty} \exp(-\psi b)\, \psi \quad \exp\left[-\tfrac{1}{2}\psi\tau''(\theta - \mu_1)^2\right] \{\sqrt{\tau\psi}\,(\theta - \mu)\}^k\, d\theta\, d\psi = \left(\frac{\tau}{\tau''}\right)^{k/2} \sum_{j=0}^{k} \binom{k}{j} (\mu_1 - \mu)^j\, \tau''^{1/2(j-1)}\, \nu_j\, \Gamma\left(\tfrac{1}{2} a_{j+1}\right) / b^{1/2 j+1} \quad (7)$$

where

$$\beta' = \beta + \tfrac{1}{2}\Sigma(t_i - \bar{t})^2 + \tau n(\bar{t} - \mu)^2/2\tau'$$

$$\tau' = \tau + n,\quad \tau_1 = (\tau - \tau')/\tau' \quad (8)$$

$$\mu' = (\tau\mu + n\bar{t})/\tau'\ ;\ \tau'' = \tau + 1;\ a_j = 2a + n + \quad (9)$$

$$b = \beta' + \tau'(x - \mu')^2/2\tau'' \quad (10)$$

$$\nu_j = \begin{cases} (j-1)(j-3) \ldots 3 & \text{when } j \text{ is even} \\ 0 & \text{when } j \text{ is odd} \end{cases} \quad (11)$$

and writing the 3-parameter t-density with a_i degrees of freedom, location parameter μ' and scale parameter σ_i, as

$$T_i = \left[+ \frac{(x - \mu')^2}{\sigma_i a_i} \right]^{1/2 a_i + 1} \Big/ \sqrt{\sigma_i a_i}\; B(\tfrac{1}{2}, \tfrac{1}{2} a_i) \quad (12)$$

with $\sigma_i = 2\tau''\beta'/\tau' a_i$, the predictive density function of a future observation x from $N(\theta, \psi)$, given a sample $\underset{\sim}{t}$, is found to be as

$$\xi(x \mid \underset{\sim}{t}) = \int^{\infty} \int \xi(\theta, \psi \mid t)\, f(x \mid \theta, \psi)\, d\theta\, d\psi$$

$$= \left(\frac{\tau'}{2\pi\tau''}\right)^{1/2} [g_0 + \tfrac{1}{6}\lambda_3\sqrt{\tau}(\tau g_3 + 3k^* g_1) + \tfrac{1}{24}\lambda_4 (\tau^2 g_4 + 6\tau k^* g_2 + 3k^{*2} g_0) + \tfrac{1}{72}\lambda_3^2(\tau^3 g_6 + 15\tau^2 k^* g_4 + 45\tau k^{*2} g_2 + 15 k^{*3} g_0)]/G \quad (13)$$

$$G = \frac{\Gamma(p_1)}{(\beta')^{p_1}}[1 + (\tfrac{1}{6}\lambda_3 A\, \Gamma(p_1 + .5)\{(p_1 + .5) A^2 + 3\tau_1\}/\Gamma(p_1) + \tfrac{1}{24}\lambda_4\{p_1(p_1+1)A^4 + 6p_1\tau_1 A^2 + 3\tau_1^2\} + \tfrac{1}{72}\lambda_3^2\{p_1(p_1+1)(p_1+2) A^6 - 15p(p_1+1)\tau_1 A^4 + 45 p_1 \tau_1^2 A^2 + 15\tau_1^3\}] \quad (14)$$

$$A = n(\bar{t}-\mu)\left(\frac{\tau}{\beta'}\right)^{1/2}/\tau',\ P_1 = (2a+n)/2,$$

$$k^* = \left(\frac{\tau}{\tau''}-1\right), \delta = \mu'-\mu \tag{15}$$

$$g_i = M_i[\delta + (x-\mu')/\tau'']^i T_i \tag{16}$$

$$M_i = [\sigma_i a_i \pi/(\beta)^{a_i+1}]^{1/2}\Gamma\left(\frac{1}{2}a_i\right) \tag{17}$$

This pdf has been used for investigating sensitivity of certain sampling inspection plans and optimum treatment allocation in decisive prediction framework[13,14]

3. RELIABILITY ESTIMATION

Failure time of a component may be considered as normally distributed random variable when its mean relatively is a much larger positive number than the standard deviation. Martz and Waller[1] have considered Bayes estimation of reliability and reliable life of a component failure time with NG as well as vague priors for the unknown mean and precision.

The reliability function for at least a period of time t_0, is given by

$$r(t_0;\theta,\psi) = 1-\Phi\{\sqrt{\psi}(t_0-\theta)\} \tag{18}$$

where $\Phi(.)$ is the cdf of standard normal distribution.

Bayes estimate of reliability, under quadratic loss is the posterior mean. Thus

$$\hat{r} = 1 - \int_{t_0}^{\infty}\int_{-\infty}^{\infty} \Phi\{\sqrt{\psi}(\theta-\psi)\}\,\xi(\theta,\psi \mid \underset{\sim}{t})\, d\theta d\psi \tag{19}$$

On rearranging the terms, as in Martz and Waller[1], we have

$$\hat{r} = \int_n \xi(x \mid \underset{\sim}{t})\,dx \tag{20}$$

where $\xi(x \mid \underset{\sim}{t})$ is the predictive density function in Eqn (13). This incomplete integral can be easily evaluated by repeated use of the integrals

$$P_k = \int_{t_0}^{\infty} g_k dx = M_k \int_{t_0}^{\infty}[\delta + (x-\mu')/\tau'']\, T_k dx$$

$$= M_k \sum_{j=0}^{k}\binom{k}{j} A_{j,k-j}\frac{\delta^j}{(\tau'')^{k-j}} \tag{21}$$

$$A_{i,2k} = \int_{t_0}^{\infty}(x-\mu')^{2k}\, T_i\, dx$$

$$= \frac{(a_i\sigma_i)^k}{2\sqrt{\pi}}\ \frac{\Gamma\left(\frac{1}{2}a_i-k\right)\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a_i\right)} \tag{22}$$

where

$$I^*_{t_0}\left(\frac{1}{2}a_i-k,\ k+\frac{1}{2}\right)$$

$t^*_0 = [1 + (t_0-\mu')^2/a_i\sigma_i]$ and

$I^*_{t_0}(.,.)$ is the incomplete beta integral (23)

After algebraic simplifications, the Bayes estimate of reliability at $t = t_0$, can be obtained as

$$\hat{r} = \left(\frac{\tau'}{2\pi\tau''}\right)^{1/2}[P_0 + \frac{1}{6}\lambda_3\sqrt{\tau}(3k^*P_1 + \tau P_3)$$

$$+ \frac{1}{24}\lambda_4(\tau^2 P_4 + 6\tau k^* P_2 + 3k^{*2}P_0) + \frac{1}{72}\lambda_3^2$$

$$(\tau^3 P_6 + 15\tau^2 k^* P_4 + 45\tau k^{*2} P_2 + 15k^{*3}P_0)]/G \tag{24}$$

Illustration 1

To study the effect of non-normality in the prior, consider the Martz and Waller[1] example (p. 116) where they consider the failure data for 22 bushing failures of a 115 kV power generator. The data is reproduced below:

8.0, 8.83, 9.50, 10.75, 11.75, 11.83, 11.92, 12.67, 12.83, 13.08, 13.50, 13.91, 14.08, 14.75, 15.0, 15.75, 16.50. 17.60, 17.83, 19.17, 20.08

These failure times exhibit good fit to normal distribution on the normal probability plot paper. Taking μ=15.4, τ=0.25, a=4.0, β=0.5 and considering number of values of λ_3 and λ_4 within the Barton-Dennis region, some values of the Bayes estimate of the reliability function at t_0=5 are presented in Table 1.

Table 1. Comparative values of $\hat{r}$ at t=5 for some EG priors

	λ				
	0	0.	0.2	0.3	0.4
0	0.998052	0.998055	0.998057	0.998060	0.998063
0.5	0.998051	0.998054	0.998056	0.998059	0.998061
1.2	0.998050	0.998052	0.998055	0.998057	0.998059
2.0	0.998049	0.998051	0.998053	0.998055	0.998058

Table 1 suggests that the Bayes estimate $\hat{r}$ is quite insensitive to ESD type of non-normality in the prior. An increase in λ_3 value tends to increase $\hat{r}$, whereas, increase in kurtosis (λ_4) decreases the $\hat{r}$ value. Thus, counter-balancing effect, may be seen. In particular, the NG prior case situation is repeated for EG prior with $(\lambda_3, \lambda_4) = (0.1, 1.2)$.

4. RELIABLE LIFE ESTIMATION

The reliable life corresponding to an $N(\theta, \psi)$ distribution, is given by

$$t_R = \psi^{-1/2} \Phi^{-1}(1-R) + \theta \tag{25}$$

where R is specified, (ref. 1, p. 436) the authors have discussed Bayes estimation of t_R when θ and ψ have either non-informative or NG priors. Under quadratic loss function, posterior mean becomes the Bayes estimate. Thus

$$E(t_R|\underline{t}) = \Phi^{-1}(1-R)\; E(\psi^{-1/2}|\underline{t}) + E(\theta|\underline{t}) \tag{26}$$

where the second expectation on the rhs of Eqn (26) is taken wrt the marginal posterior distribution of θ. However, Bansal[4] has given $\xi(\psi|\underline{t})$ and obtained $E(\theta|\underline{t})$, we have

$$E(\theta|\underline{t}) = \int_{-\infty} \theta\xi(\theta|\underline{t})\, d\theta$$

$$\mu' + [\tfrac{1}{2}\lambda_3(\tau\beta')^{1/2}\, \Gamma(p_1 - .5)\,\{(p_1 - .5)\,A^2 - \tau_1\}\; \tau' + \Gamma(p_1)\,\}$$

$$+ \tfrac{1}{12}\; \lambda_4(\tau_1 + 1)\,(\mu' - \mu)\,(2p_1A^2 + 6\tau_1 + 3)$$

$$+ \tfrac{1}{72}\; \lambda_3^2(\tau_1 + 1)\,(\mu' - \mu)\,\{2p_1\,(p_1 + 1)\,A^4 + 20p_1\,\tau_1A^2 + 30\,\tau_1^2 - 15\,\}\big/G \tag{27}$$

and after simplification

$$E\,(\psi^{-1/2}\,|\,\underline{t} = \int^{\infty} \psi^{-1/2}\,\xi(\psi\;\; \underline{t})\,d\psi$$

$$= [q_0 + \tfrac{}{2}\;\lambda_3 A\;\;(p_1 + .5)\,A^2 q_{1.5} + 3\tau_1\, q_{0.5}\,\}\;\Gamma(p_1 + .5)/\Gamma(p_1)$$

$$+ \tfrac{1}{24}\,\lambda_4\,\{\,p_1(p_1 + .5)\,A^4\,q_2 + 6p_1\tau_1\,A^2\,q_1 + 3\tau_1^2\,q_0$$

$$+ \tfrac{1}{72}\lambda_3^2\,\{p_1(p_1 + .5)\,(p_1 + 1)\,A^6 q_3 + 15p_1\,(p_1 + .5)\;\tau_1 A^4 q_2 + 45p_1\,\tau_1^2 A^2 q_1 + 15\,\tau_1^3 q_0\}\big/G \tag{28}$$

where

$$q_k = \beta' \Gamma(\,p_1 + k - \tfrac{1}{2})/\Gamma(\,p_1 + k)$$

Illustration 2

Considering the data used in Illustration 1 with R=0.8, comparative values of t_R for some EG priors are given in Table 2.

Table 2. Bayes estimate $\hat{t}_{0.8}$ for Martz-Waller data with respect to EG priors

λ_4	λ_3				
	0	0.1	0.2	0.3	0.4
0	11.585	11.584	11.574	11.556	11.528
0.5	11.586	11.585	11.576	11.558	11.532
1.2	11.587	11.586	11.578	11.562	11.537
2.0	11.588	11.587	11.580	11.565	11.542

Bayes estimate of reliable life is also quite insensitive to Edgeworth type of non-normality. It decreases with the increase in λ_3 value but increases with increase in λ_4 value. Thus, once again, a counterbalancing effect is observed.

5. CONCLUSION

It can be concluded that Bayes estimates of reliability and the reliable life, for Martz and Waller data, are quite robust to Edgeworth type of non-normality in the prior distribution of the unknown mean. Thus one could depend on NG prior and use the estimates obtained by them even when the true prior for the unknown mean was not exactly normal.

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