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A Note on the New Theory of Shock Dynamics

S.K. Singh and V.P. Singh

Centre for Aeronautical Systems Studies and Analyses, Bangalore-560 075

ABSTRACT

The new theory of shock dynamics proposed by Ravindran and Prasad has been considered with initial values taken from the similarity solution for a single conservation equation and the results have been compared with those obtained by Lax-Wendroff finite difference method.

1. INTRODUCTION

Recently a new theory of shock dynamics has been proposed[1] to predict the shock profile from a given initial data. A single conservation law also called Burgers equation without viscosity[2] was used to formulate the behaviour of shock profile and to test the validity of this new theory. A similarity solution of this model equation has been derived and taking the initial values from the similarity solution, the finite system of ordinary differential equations arising in the new theory have been solved for various degrees of truncation. The model equation has also been solved using two-step Lax-Wendroff method and results are compared.

2. DERIVATION OF INITIAL CONDITIONS FROM SIMILARITY SOLUTION

Let us consider a single conservation law

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \tag{1}$$

Let

$$x = X(t) \tag{2}$$

be a surface of discontinuity in the solution field of Eqn (1). We assume that the discontinuity front can be expressed as

$$X(t) = C\,t^a \tag{3}$$

where C and a are arbitrary constants.

Introducing a non-dimensional variable

$$\xi = \frac{x}{X(t)} \tag{4}$$

and letting

$$u = \frac{X(t)}{t}\,V(\xi) \tag{5}$$

the model Eqn (1) reduces the first order homogeneous ordinary differential equation (ODE)

$$\frac{dV}{d\xi} = \frac{(1-a)V}{V-a\xi} \tag{6}$$

The solution of Eqn (6) is given by

$$V^a(\xi - V)^{1-a} = K \tag{7}$$

where K is a constant to be determined from the initial conditions.

Assuming the value of u ahead of shock to be negligible compared to that behind it, (i.e., u_0) for all t, the jump condition for Eqn (1) reduces to[3]

$$\frac{dX}{dt} = \frac{u_0}{2} \tag{8}$$

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hence at ξ (i.e at shock front), $V(1 = 2a$ and

$$K = (2a)^a - 2a)^1 \quad (9)$$

The solution given by Eqns (7) and (9) is valid for any arbitrary assigned value of a. It has been observed in the case of gas dynamics[3] that $a = 2/3$ for plane shock. So in the present study, a typical case in which $a = 2/3$ has been taken.

For this value of a, Eqn (7) reduces to the cubic equation

$$V - \xi V^2 - \frac{16}{27} = 0 \quad (10)$$

which admits only one real solution given by

$$V = \frac{1}{3}(\xi^3 + 8 + 4\sqrt{\xi^3 + 4})^{1/3} + \frac{1}{3}(\xi^3 + 8 - 4\sqrt{\xi^3 + 4})^{1/3} + \frac{\xi}{3} \quad (11)$$

For any preassigned value of t, the Eqns (3), (5) and (11) give the initial condition at any point behind the shock front. The initial shock profile corresponding to Eqn (11) is shown in Fig. 1.

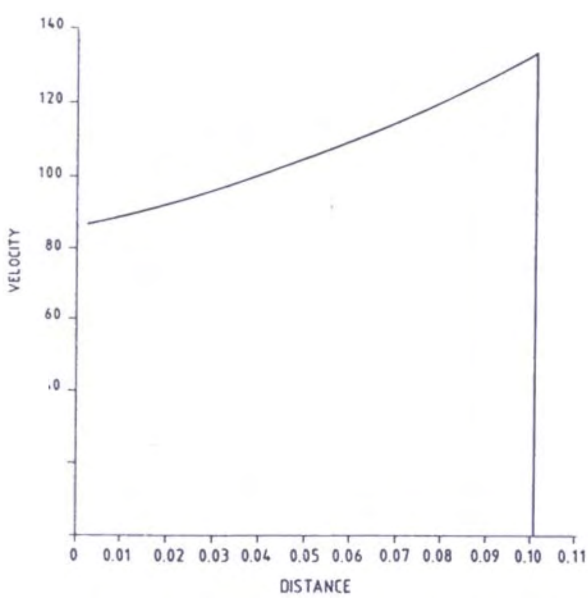

Figure 1. Initial shock profile from the similarity solution of the single conservation equation for $C = 10$, $t_0 = 0.001$, and $a = 2/3$.

3. SOLUTION OF FINITE SYSTEM OF COMPATIBILITY CONDITIONS

The termination of the infinite system of the compatibility conditions[1] gives the following system of a finite number of first order ODE[3] (by putting $V_{N+1} = 0$ in the $(N+1)$th equation):

$$\frac{dX}{dt} = \frac{1}{} u_0$$

$$\frac{du_0}{dt} = \frac{}{2}$$

$$\frac{dv_i}{dt} = -\frac{(i+1)}{2} u_0 v_{i+1} - \frac{(i+1)}{2} \Sigma_{i}^{i} v_j v_{i-j+1};$$

$i = 1,2,3$

$$\frac{dv_N}{dt} = -\frac{(N+1)}{2} \Sigma_{j=1}^{N} v_j v_{N-j+1} \quad (12)$$

where $v_i = u_i(t)/i!$; $u_i(t)$ being the ith spatial derivative of u at the shock $x = X(t)$. The approximate solution is given by

$$u(x,t) = u_0(t) + \Sigma_{i=1}^{N} v_i(t) \{x - X(t)\}^i, x < X(t) \quad (13)$$

Equation (12) has been integrated with initial values from Eqn (11) using Runge-Kutta-Gill method and the results obtained are shown in Fig. 2.

4. NUMERICAL SOLUTION USING LAX-WENDROFF METHOD

In this section, Eqn (1) has been numerically integrated using Lax-Wendroff scheme. The two-step Lax-Wendroff scheme for Eqn (1), with artificial viscosity can be written as[5]

$$u_{m+1/2}^{n+1/2} = \frac{1}{2}(u_{m+1}^n + u_m^n) - \frac{p}{4}\left\{(u^2)_{m+1}^n - (u^2)_m^n\right\}$$

$$u_m^{n+1} = u_m^n - \frac{p}{2}\left\{(u^2)_{m+1/2}^{n+1/2} - (u^2)_{m-1/2}^{n+1/2}\right\}$$

$$\frac{p}{2}\left\{(u_{m+1}^n - u_m^n)|u_{m+1}^n - u_m^n| - (u_m^n - u_{m-1}^n)|u_m^n - u_{m-1}^n|\right\} \quad (14)$$

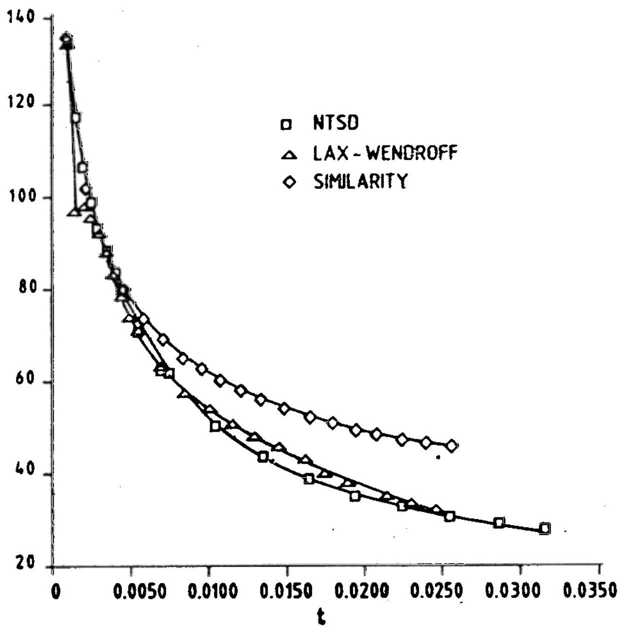

Figure 2. $u-t$ curves for single conservation law by various methods.

for $0 \leq m \leq M$, $n = 0, 1, 2$ and $p = \frac{\Delta t}{\Delta x}$

together with the numerical boundary condition at $m = M$,

$$u_M^{n+1} = u_M^n - \frac{p}{2}\left\{(u^2)_M^n - (u^2)_{M-1}^n\right\} \qquad (15)$$

Results of numerical integration are shown in Fig. 2.

5. NUMERICAL RESULTS AND DISCUSSION

Choosing $C = 10.0$, $t_0 = 0.001$, $\xi = 0.99$, the finite system of compatibility conditions has been solved for $N = 6$ (Fig. 2). A single conservation law was also solved numerically using two-step Lax-Wendroff method with $p = -0.005$ using artificial viscosity to reduce the oscillations behind the shock. This solution is plotted along with the solutions obtained in Section 3 with $N = 6$, and also with the similarity solution. It is seen that the solutions obtained by the new method and that by Lax-Wendroff method exhibit good agreement whereas the similarity solution shows considerable deviation as it does not hold for larger values of t (Fig. 2).

It was also observed that the finite difference method consumes much more computer time for finer grid spacing as compared to that taken by method of new theory of shock dynamics. Also it gives oscillations for points, on and just behind the shock front which can only be minimised using artificial viscosity but cannot be eliminated. It is observed from the above discussions that the new theory of shock dynamics gives a reasonably satisfactory alternative to the numerical scheme in terms of accuracy as well as computation time.

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REFERENCES

1. Ravindran, R. & Prasad, P. A new theory of shock dynamics, Part I : Analytic considerations. *Appl. Math. Lett.*, 1990, 3(2), 77-81.
2. Smoller, Joel. Shock waves and reaction-diffusion equations. Springer-Verlag, New York, 1983.
3. Prasad, P. & Ravindran, R. Partial differential equations. Wiley Eastern Ltd., New Delhi, 1987.
4. Sedov, L.I. Similarity and dimensional methods in mechanics, Ed.1. Mir Publishers, Moscow, 1982.
5. Richtmyer, R.D. & Morton, K.W. Difference methods for initial value problems, Ed.2. Interscience Publishers, New York, 1967.